Dr. Z.'s Calc4 Lecture 1 Handout: Introducing Differential Equations

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Section 1: Direction Fields

A general **first order** *differential equation* looks like

$$\frac{dy}{dt} = f(y,t) \quad ,$$

where $f(t,y)$ is (usually) a function of **both** t (usually time), and y (the value of the desired function at time t, or graphically, the elevation).

It often has an *initial condition*: $y(0) = y_0$ (or, more generally, $y(t_0) = y_0$), for some number y_0. we often write y' instead of $\frac{dy}{dt}$.

Here are some examples

$$y' = t + y \quad , \quad y(0) = 5 \quad ;$$

$$y' = ty \quad , \quad y(0) = 5 \quad ;$$

$$y' = \cos(ty) \quad , \quad y(0) = \pi \quad .$$

Such a differential equation is a **puzzle**. For example the differential equation $y' = y + t$ means

I am a certain function of t, let's call me $y(t)$. My derivative (rate of change) at any time t in the future is **exactly** equal to t plus my value at that very same time ($y(t)$), who could I possibly be? How do I look like?

If you also have the initial condition $y(0) = 5$, for example, then there is an additional clue.

"In addition, at $t = 0$, my value is exactly 5". Who am I?

For quite a few (but by no means for all!) differential equations, people (and Maple!) know how to find an **exact formula** for $y(t)$. For example, for the diff.eq. $y' = y + t, y(0) = 5$, you type

```
dsolve({diff(y(t),t) = y(t) + t, y(0) = 5}, y(t));
```

and you get immediately

```
y(t)=-1-t+6 exp(t)      ,
```

and if you want a plot, all you have to do is type:

```
plot(op(2,%),t);
```

But for many diff.eqs. there is no way to find a **formula** for the solution. For these one can do it graphically, by drawing a **direction field**.

To solve a diff.eq. of the form $y' = f(t, y)$ draw a **tiny arrow** of slope $f(t, y)$ starting at the point (t, y) for as many points $(t.y)$ as you can. Then starting at $(0, y_0)$ “follow the arrows” and you would get the graph of the solution.

If you are not given an initial condition, then there are *infinitely many* solutions! If you are lucky, then amongst them would be a very simple solution, a **horizontal line**, aka as a *constant function*, $y(t) = c$ for some constant c. Since for such constant functions $y' = 0$ always, to find out whether you are lucky, you have to solve the (non-differential!) equation

$$f(c, t) = 0 \quad ,$$

and see whether you can solve for c that does not depend on t. This usually happens for the special case that $f(t, y)$ **does not** depend on t, i.e. the slope of the function $y(t)$ only depends on the elevation, but not on the time! Such equations are called **autonomous**, and their format it

$$y' = F(y)$$

for some function $F(y)$ of **only** y, and then the constant solutions are those c for which $F(c) = 0$.

Such solutions, of the form $y(t) = c$ are called **equilibrium** solutions, since, if in luck, in “the long run” all solutions **converge** to them as time, t, gets larger and larger. Otherwise solutions *diverge from it.*

Problem 1.1.1: Draw a direction field for the given differential equations. Based on the direction field, determine the behavior of y as $t \to \infty$

a: $y' = 6 - 2y$

b: $y' = 6 + 2y$

Solution of 1.1.1.a: Solving $6 - 2y = 0$, we get the constant function $y = 3$. If $y > 3$ then the slope is **negative** so the function is going down. If $y < 3$ then the slope is **positive**, so the function is going up. So in the long-run all solutions get closer and closer to the horizontal line $y = 3$.

Ans. to 1.1.1.a: $y \to 3$ as $t \to \infty$

Solution of 1.1.1.b: Solving $6 + 2y = 0$, we get the constant function $y = -3$. If $y > -3$ then the slope is **positive** so the function is going up. If $y < -3$ then the slope is **negative**, so the function is going down. So in the long-run all solutions **diverge from** the horizontal line $y = -3$.

Ans. to 1.1.1.b: y diverges from -3 as $t \to \infty$

Problem 1.1.2: Draw a direction field for the differential equation

$$y' = (1 - y)(y - 2) \quad .$$

Based on the direction field, determine the behavior of y as $t \to \infty$. If it depends on the initial condition, then state the behavior accordingly.

Solution of 1.1.2: Using algebra, we solve the **algebraic** equation $(1-y)(y-2)=0$. We get the **two** solutions $y=1$ and $y=2$. If $y>2$ then the slope is **negative** so the function is going down. If $1<y<2$ then the slope is **positive**, so the function is going up. So in the long-run all solutions for which $y_0>1$ converge to the line $y=2$. If $y<1$ (e.g. $y=0$) then the slope is **negative** and the function is going down to $-\infty$.

Ans. to 1.1.2: For $y>1$, $y\to 2$ as $t\to\infty$. For $y<1$, y diverges from $y=1$.

1.2: What does it mean to be an explicit solution of a DiffEq

Problem 1.2.1: Verify that $y(t)=e^t+e^{2t}$ is a solution of the initial value differential equation

$$y''-3y'+2y=0 \quad , \quad y(0)=2 \quad , \quad y'(0)=3 \quad .$$

Solution to 1.2.1: Using calculus,

$$y(t)=e^t+e^{2t} \quad , \quad y'(t)=e^t+2e^{2t} \quad , \quad y''(t)=e^t+4e^{2t} \quad .$$

Plugging-in,

$$y''(t)-3y'(t)+2y(t)=e^t+4e^{2t}-3(e^t+2e^{2t})+2(e^t+e^{2t})=e^t(1-3+2)+e^{2t}(4-6+2)=0 \quad .$$

So the diff.eq. is OK. Now,

$$y(0)=e^0+e^0=1+1=2 \quad , \quad y'(0)=e^0+2e^0=1+2=3 \quad .$$

So the initial conditions are OK.

Problem 1.2.2: Verify that $y(t)=t^2$ is a solution of the initial value differential equation

$$y'(t)^3-ty(t)=7t^3 \quad , \quad y(1)=1 \quad .$$

Solution to 1.2.2: Using calculus

$$y(t)=t^2 \quad , \quad y'(t)=2t \quad ,$$

Using algebra

$$(y'(t))^3-t\,y(t)=(2t)^3-t(t^2)=8t^3-t^3=7t^3 \quad .$$

So the diff.eq. is OK. Now

$$y(1)=1^2=1 \quad ,$$

so the initial condition is OK too.

1.3: Classification of Differential Equations

If the function we are looking for, $y(t)$, and all its derivatives, $y'(t), y''(t), \ldots$ that show up, are all by themselves, (i.e. not raised to a power, or inside a function!,or multiplied by each other), then the diff.eq. is **linear**. The highest derivative that shows up is the **order**.

The **format** of a *homogeneous* linear diff.eq. or order r is

$$f_r(t)y^{(r)}(t) + f_{r-1}(t)y^{(r-1)}(t) + \ldots + f_0(t)y(t) = 0,$$

where $f_r(t), f_{r-1}(t), \ldots f_0(t)$ are functions of t. The format of an *inhomogeneous* linear diff.eq. or order r is

$$f_r(t)y^{(r)}(t) + f_{r-1}(t)y^{(r-1)}(t) + \ldots + f_0(t)y(t) = R(t) \quad ,$$

where $R(t)$ is yet another function of t.

If a diff.eq. does not have such a format, e.g. $y'(t)y(t) + t = 5$ or $y'(t) + y(t)^2 = 0$ then it is **non-linear**.

Problem 1.3.1: For each of of the following diff.eq. state whether there are linear or non-linear, and find the order.

a: $y''(t) + 6y'(t) + y(t) = 7 \quad ,$

b: $y'''(t) + (\sin t)\, y'(t) + y(t) = 7 \quad ,$

c: $y''(t)^2 + y'(t) + y(t) = 0 \quad ,$

d: $y^{(4)}(t) + y'(t)y(t) = 6 \quad .$

Solution of 1.3.1: In (a) the highest derivative that shows up is the second, so it is a second-order diff.eq. Since each of $y(t), y'(t), y''(t)$ are by themselves, this diff.eq. is **linear**.

In (b) the highest derivative that shows up is the third ($y'''(t)$), so it is a third-order diff.eq. Since each of $y(t), y'(t), y''(t)$ are by themselves it is **linear**. Note that even though the coeff. of $y'(t)$ is not a number (it is a function, $\sin t$), the diff.eq. is still linear (but it is **not** constant-coefficients).

In (c) the highest derivative that shows up is the second ($y''(t)$), so it is a second-order diff.eq. Since $y''(t)$ is squared, it is **non-linear**.

In (d) the highest derivative that shows up is the fourth ($y^{(4)}(t)$), so it is a fourth-order diff.eq. Since $y(t)$ and $y'(t)$ are multiplied by each other, it is **non-linear**.

Ans. to 1.3.1: (a) is a second-order linear diff.eq; (b) is a third-order linear diff.eq. ; (c) is a second-order non-linear diff.eq. ; (d) is a fourth-order non-linear diff.eq.