MATH 251 (22,23,24) Brief indications and answers to Dr. Z. Second Chance Club for Exam 2 Worksheet

Problem 1a. Compute the line integral

$$\int_C xe^{xyz}\,dx + ye^{xyz}\,dy + ze^{xyz}\,dz \quad ,$$

over the path

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Explain!

Comment: Thie problem was assigned by mistake. The point is to use the Fundamental Theorem of Line Integrals. But taking the curl indicates that it is **not** conservative. Of course one can do it directly, getting a messy t integral, that even Maple can't do exactly, but using `evalf(Int(MONSTER, t=0..1);` one can get the answer that I emailed you.

Problem 1b. Compute the line integral

$$\int_C (4x^3y^2+1)\,dx + (2x^4y+1)\,dy \quad ,$$

over the path

$$\mathbf{r}(t) = \langle \sin t^2, \cos t^2 \rangle \quad , \quad 0 \le t \le \sqrt{\pi/2} \quad .$$

Explain!

Skecth of Sol. to 1b: Here it is conservative, since $P_y = Q_x$, the potentail function is $f(x,y) = x^4y^2+x+y$ (you do it!). The start is $(0,1)$ the end is $(1,0)$ and $f(1,0)-f(0,1) = 0$. **Ans.**: 0.

Problem 2a:

Change the order of integration

$$\int_0^1 \int_0^{e^x} f(x,y)dy\,dx$$

Sol. of 2a: This is type I. If you sketch it (you do it), the area in question is above the line segment $0 < x < 1$ on the x-axis and **under** the curve $y = e^x$.

$$D = \{(x,y) : 0 < x < 1, 0 < y < e^x\} \quad .$$

From the type II perspective, the projection on the y-axis is the line segment $0 < y < e$ and a typical horozontal cross-section is from where $y = e^x$ in other words $x = \ln y$ all the way to the vertical line $x = 1$. So the type II description is

$$D = \{(x,y) : 0 < y < e \, , \ln y < x < e\} \quad .$$

So the integral is now

$$\int_0^e \int_{\ln y}^e f(x,y) dx \, dy \quad .$$

Problem 2b:

Change the order of integration

$$\int_0^\pi \int_0^{\sin x} f(x,y) dy \, dx$$

Sol. of 2b. The type I description is

$$D = \{(x,y) : 0 < x < \pi, 0 < y < \sin x\} \quad .$$

If you plot it, this is the area above $0 < x < \pi$ on the x-axis and under $y = \sin x$. The projection on the y axis is the line segment $0 < y < 1$ and a typical horizontal part starts at $x = \sin^{-1} y$ and ends at $x = \pi - \sin^{-1} y$. So the type II description is

$$D = \{(x,y) : 0 < y < 1, \sin^{-1} y < \ x \ < \pi - \sin^{-1} y\} \quad .$$

And the integral is

$$\int_0^1 \int_{\sin^{-1} y}^{\pi - \sin^{-1} y} f(x,y) dx \, dy$$

Problem 2c:

Change the order of integration

$$\int_0^1 \int_{e^y}^e f(x,y) dx \, dy$$

Sol. or 2c: Now it is in Type II format

$$D = \{(x, y) : 0 < y < 1, e^y < x < e\} \quad .$$

If you plot it, the projection on the x-axis is the interval $0 < x < e$ and any horizontal cross-section starts where $x = e^y$ and ends where $x = e$. But $x = e^y$ is the same as saying $y = \ln x$, so the required region is under $y = \ln x$, and the type I description i

$$D = \{(x, y) : 0 < x < e, 0 < y < \ln x\} \quad ,$$

giving

$$\int_0^e \int_0^{\ln x} f(x, y) dy\, dx$$

Problem 3a. Find the equation of the tangent plane at the point $(1, 2, 4)$ to the surface given parametrically by

$$x(u, v) = u^2 \quad , \quad y(u, x) = uv \quad , \quad z(u, v) = v^2 \quad , \quad -\infty < u < \infty \quad , \quad -\infty < v < \infty \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

Sketch: The relevant point is $(u, v) = (1, 2)$, You find $\mathbf{r}_u$ and $\mathbf{r}_v$,plug-in $u = 1$ and $v = 2$. Take the cross product $\mathbf{r}_u(1, 2) \times \mathbf{r}_v(1, 2)$ get the normal vector $\mathbf{N}$ and simplify $\mathbf{N}.\langle x - 1, y - 2, z - 4\rangle = 0$, and finally use algebra to get it in explicit form.

Problem 3b. Find the equation of the tangent plane at the point $(-1, -1, 2)$ to the surface given parametrically by

$$x(u, v) = u^3 \quad , \quad y(u, x) = v^3 \quad , \quad z(u, v) = -2uv \quad , \quad -\infty < u < \infty \quad , \quad -\infty < v < \infty \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

Sol. to 3b: From the first two equations you get $u = -1$ and $v = -1$ but then $z = -2$ **not** $z = 2$. So the point does **not** lie on the surface, so the answer is N/A.

Comment: This was a typo. I meant the point to be $(-1, -1, -2)$. But if it comes in an exam you would get full credit. Don't try to "correct" the question.

Problem 4a Let $f(x, y, z) = \sin(x + y^2 + z^3)$, and let

$$\mathbf{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

Let C be the curve

$$r(t) = \langle t, t^2, t^3 \rangle \quad , \quad 0 \leq t \leq 3 \quad .$$

Find the value of the line-integral

$$\int_C \mathbf{F}.d\mathbf{r} \quad .$$

Explain! Just giving the answer will give you no credit.

Sol. of 4a: This is really easy. $\mathbf{F} = grad(f)$ so you don't have to find the potential function, it is given to to you. By the Fundamental Theorem of Line Integrals the answer is $f(End) - f(Start)$. The starting point is $(0, 0, 0)$ the ending point is $(3, 9, 27)$ so the answer is $\sin(3 + 9^2 + 27^3) - \sin(0) = \sin 19767$

Problem 4b Let $f(x, y) = e^{cos x + 3 \sin y}$, and let

$$\mathbf{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$$

Let C be the curve

$$r(t) = \langle \sin 2t, \cos t \rangle \quad , \quad 0 \leq t \leq \pi \quad .$$

Find the value of the line-integral

$$\int_C \mathbf{F}.d\mathbf{r} \quad .$$

Explain! Just giving the answer will give you no credit.

Sketch: Same way! Answer is 0.

Problem 5a Evaluate the triple integral

$$\int_R (x + y)(x^2 + y^2 + z^2)^2 \, dx \, dy \, dz \quad ,$$

where R is the region in 3D space given by

$$\{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 1 \quad , \quad x \geq 0, y < 0, z < 0\} \quad .$$

Sketch of Sol. of 5a : You translate to spherical language. The only challenging part are the limits. Of course ρ goes from 0 to 1. Since $z < 0$ it follows that $\pi/2 < \phi < \pi$, since $x > 0, y < 0$ the projection on the equatoirial plane (i.e. the xy-plane) is the **fourth** quadrant where $\frac{3\pi}{2} < \theta < 2\pi$. The answer turns out to be 0.

Problem 5b Evaluate the triple integral

$$\int_R z(x^2 + y^2 + z^2)\, dx\, dy\, dz \quad ,$$

where R is the region in 3D space given by

$$\{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 1 \quad , \quad y < 0\} \quad .$$

Sketch of Sol. of 5b : You translate to spherical language. The only challenging part are the limits. Of course *rho* goes from 0 to 1. Since z is no restricted, ϕ gets its full scope $0 < \phi < \pi$. Since $y < 0$ we are talking about the lower-half-plane (under the x-axis) so the $\pi < \theta < 2\pi$. The answer turns out to be 0.

Problem 5c Evaluate the triple integral

$$\int_R (z - x)\, dx\, dy\, dz \quad ,$$

where R is the region in 3D space given by

$$\{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 8\} \quad .$$

Sketch of Sol. of 5c : You translate to spherical language. The only challenging part are the limits. Of course ρ goes from 0 to $\sqrt{8}$. Since there are no restictions on x, y, z ϕ and θ take their default limits of integration $[0, \pi]$ and $[0, 2\pi]$ respectively. The answer turns out to be 0.

Problem 6a Convert the integral to polar coordinates, do not evaluate.

$$\int_{-3}^{0} \int_{0}^{\sqrt{9-x^2}} (x^2 + y)\, dy\, dx$$

Sketch of Sol. of 6a. You convert to polar langauge. The region is the left half of the disk $x^2 + y^2 < 9$. r goes from 0 to 3, while θ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

Problem 6b Convert the integral to polar coordinates, do not evaluate.

$$\int_0^4 \int_{-\sqrt{16-x^2}}^0 (x^2 + y)\, dy\, dx$$

Sketch of Sol. of 6b. You convert to polar langauge. The region is the bottom half of the disk $x^2 + y^2 < 9$. r goes from 0 to 4, while θ goes from π to 2π .

Problem 6c Convert the integral to polar coordinates, do not evaluate.

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} (x^3 + y^2)\, dy\, dx$$

Comment: This problem was assigned by mistake. It is too hard for the Final. I meant to ask about

$$\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} (x^3 + y^2)\, dy\, dx$$

For this simplified problem (the one I intended) the region is one-eigth of the disk. namely one where θ goes from 0 to $\pi/4$. The answer to this version is $\frac{\pi}{24}$. The original problem is more challenging, you have to use trig, and the answer is $\frac{7}{90} + \frac{\pi}{12}$.

Problem 7a: Decide whether the following limit exists. If it does, find it, if not, explain!

$$\lim_{(x,y)\to(1,3)} \frac{x-1}{y-3} \quad ,$$

Problem 7b: Decide whether the following limit exists. If it does, find it, if not, explain!

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+2z}{2x+y+z} \quad ,$$

Sketch of Sol. Both limits do not exist. Pick two random lines that pass throgh the point in question and show that you get different answers.

Problem 8a Compute the line integral $\int_C f \, ds$ where

$$f(x, y, z) = xy^2 + yz^2 + z$$

and C is the line segment starting at $(0, 0, 0)$ and ending at $(1, 1, -1)$

Sketch of Sol. to 8a: You do it directly. First you have to figure out the parametric representation of the line segment. It is $\mathbf{r}(t) = \langle t, t, -t \rangle$, $0 < t < 1$.

Problem 8b Compute the line integral $\int_C f \, ds$ where

$$f(x, y) = x + y$$

and C is the upper circle $\{(x, y) : x^2 + y^2 = 1, y > 0\}$.

Sketch of Sol. to 8a: You do it directly. First you have to figure out the parametric representation of the upper circle. It is $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 < t < \pi$.

Problem 9a Compute the vector-field surface integral $\int \int_S \mathbf{F} . d\mathbf{S}$ if $\mathbf{F}$ is

$$\mathbf{F} = \langle x + z, y + z, -x \rangle \quad ,$$

and S is the oriented surface

$$z = 9 - x^2 - y^2 \quad , x < 0, y < 0, z \geq 0$$

with **upward pointing** normal.

Sketch of So. to 9a: This is similar to the exam problem, except that it is in the third-quadrant.

Problem 9b Compute the vector-field surface integral $\int \int_S \mathbf{F} . d\mathbf{S}$ if $\mathbf{F}$ is

$$\mathbf{F} = \langle x + z, y + z, -x \rangle \quad ,$$

and S is the oriented surface

$$z = 9 - x^2 - y^2 \quad , 0 < x < 1, 0 < y < 1, z \geq 0$$

with **downward pointing** normal.

Sketch of So. to 9b: This is similar to the exam problem, except that it is easier. Now you are specifically given the projection on the xy plane that is $\{(x, y) : 0 < x < 1, 0 < y < 1\}$

Problem 10a Find the **maximum value** of the function $f(x, y, z) = xyz$ on the plane $2x + y + z = 4$

Sketch of Sol. to 10a: This is similar to the exam problem, except that now we are asked for the **value** not the point. The answer turns out to be $\frac{32}{27}$.

Problem 10b Find the point on the plane $2x + y + z = 4$ where $f(x, y, z) = xy^2z$ is as large as possible. (You can use Maple)

Sketch of Sol. to 10b:

$$grad(f) = \langle y^2z, 2xyz, xy^2 \rangle \quad , \quad grad(g) = \langle 2, 1, 1 \rangle \quad ,$$

Spelling out $grad(f) = \lambda grad(g)$ and adding the constraint we get

$$\{y^2z = 2\lambda, 2xyz = \lambda, xy^2 = \lambda, 2x + y + z = 4\} \quad .$$

Going into Maple, and typing

```
solve(y**2*z=2*L,2*x*y*z=L,x*y**2=L,2*x+y+z=4,x,y,z,L);
```

gives you (in addition to two trivial solutions)

```
L = 2, x = 1/2, y = 2, z = 1
```

Discarding the L we get that the desired point is $(\frac{1}{2}, 2, 1)$.