Solutions to the "QUIZ" for Lecture 5

1, Find the curvature for

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\mathbf{j} + t\,\mathbf{k} \quad .$$

Solution: In conventional notation (that I find easier)

$$\mathbf{r}(t) = \langle \sin t\,,\, \cos t\,,\, t\,\rangle \quad .$$

The formula for the curvature is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

We have

$$\mathbf{r}'(t) = \langle \cos t\,,\, -\sin t\,,\, 1\,\rangle \quad .$$

$$\mathbf{r}''(t) = \langle -\sin t\,,\, -\cos t\,,\, 0\,\rangle \quad .$$

Taking their cross-product:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} \quad =$$

$$\mathbf{i}\begin{vmatrix} -\sin t & 1 \\ -\cos t & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \cos t & 1 \\ -\sin t & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix}$$

$$= \mathbf{i}\cos t - \mathbf{j}(\sin t) + \mathbf{k}(-\cos^2 t - \sin^2 t)$$

$$= (\cos t)\,\mathbf{i} + (-\sin t)\,\mathbf{j} + (-1)\,\mathbf{k} \quad ,$$

using the famous trig. identity

$$\cos^2 t + \sin^2 t = 1 \quad .$$

Translating to conventional notation we have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle \cos t\,,\, -\sin t\,,\, -1\rangle \quad .$$

So the **magnitude** is:

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (-1)^2} = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.$$

We also have

$$|\mathbf{r}'(t)| = |\langle \cos t\,,\, -\sin t\,,\, 1\,\rangle| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \quad .$$

So, finally:

$$\kappa(t) = \frac{\sqrt{2}}{\sqrt{2}^3} = \frac{1}{2} \quad .$$

Ans.: $\frac{1}{2}$.

Comments: Unfortunately, only about %40 of the students got it completely. Quite a few people got correct, but unsimplified, answers. For example $\sqrt{2}/\sqrt{2}^3$. Many people were unable to replace

$$\cos^2 t + \sin^2 t$$

by 1, getting very complicated-looking answers. Most people started it correctly, but sooner or later messed up. Either getting the wrong sign (not keeping track that minus times minus is plus and minus times minus times minus is minus etc.), or in simplifying the trig.

Please review your elementary trig and elementary algebra! One wrong sign can turn an easy problem to a hard one (and not the one that you need to do). If things get too complicated it means that you are probably on the wrong-track.

2.: Find the velocity, acceleration, and speed of a particle with the given position function.

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 5\,\mathbf{k} \quad .$$

Solution:

$$\mathbf{v}(t) = t'\,\mathbf{i} + (t^2)'\,\mathbf{j} + 5'\,\mathbf{k} = 1\,\mathbf{i} + (2t)\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{i} + 2t\,\mathbf{j}$$

$$\mathbf{a}(t) = 1'\,\mathbf{i} + (2t)'\,\mathbf{j} = 0\,\mathbf{i} + 2\,\mathbf{j} = 2\,\mathbf{j} \quad .$$

The speed is

$$|v(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1^2 + 4t^2} \quad .$$

Comments: About %90 of the people got it right. Quite a few people **oversimplified** by replacing $\sqrt{1^2 + 4t^2}$ by $1 + 2t$ using the "rule" $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. This is **WRONG**. Please don't do it again!