Solution to the "QUIZ" for Lecture 22

Evaluate the surface integral $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and oriented surface S.

$$\mathbf{F}(x,y,z) = \langle xy\,,\, yz\,,\, zx\,\rangle \quad ,$$

and S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the square $0 \le x \le 1\,,\, 0 \le y \le 1$ and has upward orientation.

Solution

The easiest way is to use the formula for **explicitly**-defined surfaces $z = g(x,y)$, and a vector field $\mathbf{F} = \langle P, Q, R\rangle$:

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R)\, dA \quad .$$

where D is the "floor", in this problem $0 \le x, y \le 1$.

In this problem $P = xy, Q = yz, R = zx$, $g(x,y) = 1 - x^2 - y^2$, so we have

$$\int\int_D (-xy(-2x) - yz(-2y) + xz)dA = \int\int_D (2x^2y + 2y^2z + xz)dA \quad .$$

In addition we **must** replace z by $g(x,y)$ ($1 - x^2 - y^2$ in this problem). So we have

$$\int\int_D (2x^2y + 2y^2z + xz)dA \;=\; \int\int_D (2x^2y + 2y^2(1 - x^2 - y^2) + x(1 - x^2 - y^2))\, dA \quad .$$

Doing the tedius algebra we get

$$\int_0^1 \int_0^1 (2x^2y + 2y^2 - 2y^2x^2 - 2y^4 + x - x^3 - xy^2)\, dx\, dy \quad .$$

Now you do the inside integral, getting someting in y, and then the outside integral, and it comes out to $\frac{83}{180}$.

Ans. $\frac{83}{180}$ (type number).