Solutions to the "QUIZ" for Lecture 19

1. Determine whether or not the vector field

$$F(x,y,z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

Sol. Here $\mathbf{F} = \langle y^2z^3\,,\,2xyz^3\,,\,3xy^2z^2\rangle$,

so $F_1 = y^2z^3$, $F_2 = 2xyz^3$, $F_3 = 3xy^2z^2$.

$\frac{\partial F_1}{\partial y} = 2yz^3$, $\frac{\partial F_2}{\partial x} = 2yz^3$,

so $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ and the first condition is OK. Also

$\frac{\partial F_1}{\partial z} = 3y^2z^2$, $\frac{\partial F_3}{\partial x} = 2y^2z^2$,

so $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ and the second condition is OK. Finally

$\frac{\partial F_2}{\partial z} = 6xyz^2$, $\frac{\partial F_3}{\partial y} = 6xyz^2$,

so $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$ and the third condition is OK.

Now it is time to find the potential function f .

From

$$\frac{\partial f}{\partial x} = F_1 \quad ,$$

We get

$$\frac{\partial f}{\partial x} = y^2z^3 \quad .$$

Integrating with respect to x, we get

$$f(x,y,z) = \int y^2z^3\,dx = xy^2z^3 + g(y,z) \quad .$$

Using

$$\frac{\partial f}{\partial y} = F_2 \quad ,$$

in other word

$$\frac{\partial f}{\partial y} = 2xyz^3 \quad ,$$

We get:

$$2xyz^3 + g_y(y,z) = 2xyz^3 \quad .$$

This meant

$$g_y(y,z) = 0 \quad .$$

Integrating with respect to y, we get:

$$g(y, z) = \int 0\, dy = 0 + h(z) \quad .$$

Going back above, we have:

$$f(x, y, z) = xy^2 z^3 + h(z) \quad .$$

Using

$$\frac{\partial f}{\partial z} = F_3 \quad ,$$

$$\frac{\partial f}{\partial z} = 3xy^2 z^2 \quad ,$$

we get

$$3xy^2 z^2 + h'(z) = 3xy^2 z^2$$

Doing the algebra, we get

$$h'(z) = 0 \quad .$$

Integrating, we get

$$h(z) = \int 0\, dz = 0 + C$$

So

$$f(x, y, z) = xy^2 z^3 + C \quad .$$

But you don't have to write the $+C$.

Ans. $f(x, y, z) = xy^2 z^3$ (type: multivariable function).

2. Show that the line integral

$$\int_C 2xy^2\, dx \;+\; 2x^2 y\, dy \quad ,$$

is independent of the path C, and evaluate it if C is *any* path from $(1, 0)$ to $(0, 1)$.

Sol. Here $\mathbf{F} = \langle 2xy^2 \,, 2x^2 y \rangle$. So $F_1 = 2xy^2, F_2 = 2x^2 y$. Since $(F_1)_y = 4xy, (F_2)_x = 4xy$, $(F_1)_y = (F_2)_x$ so this is a **conservative** vector field.

Let's find a potential function.

$$f(x, y) = \int 2xy^2\, dx = x^2 y^2 + g(y) \quad ,$$

So

$$\frac{\partial f}{\partial y} = 2x^2 y + g'(y) \quad .$$

Using

$$\frac{\partial f}{\partial y} = 2x^2 y \quad .$$

We get

$$2x^2y + g'(y) = 2x^2y$$

So $g'(y) = 0$, and so $g(y) = C$, and

$$f(x, y) = x^2y^2 \quad .$$

The line-integral (regadless of the path) is always

$$f(END) - f(START) \quad .$$

The question tells you that the path starts at $(1, 0)$ and ends at $(0, 1)$. So the answer is:

$$f(0, 1) - f(1, 0) = 0^2 \cdot 1 - 1^2 \cdot 0 = 0 \quad .$$

Ans. 0 (type number).