Solutions to the "QUIZ" of Lecture 15

1. Use polar coordinates to compute the double integral

$$\int\int_D xy\,dA \quad ,$$

where

$$D = \{(x,y)\,|\,x^2+y^2 \leq 1\,,\, x \geq 0\,,\, y \geq 0\,\} \quad .$$

Sol.: The **region** is the quarter-circle, center origin, radius 1, in the first quadrant. Its **polar coordinates** description is:

$$\{(r,\theta)\mid 0 \leq \theta \leq \pi/2 \quad , \quad 0 \leq r \leq 1\}$$

doing the **translation** we get

$$\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta) r\,dr\,d\theta$$

This is the **most important** part. From here, Maple can do it. Doing it by hand, we have

$$\int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta = \left(\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

$$= \left(\int_0^{\pi/2}\frac{\sin 2\theta}{2}\,d\theta\right)\left(\int_0^1 r^3\,dr\right) = \left(\frac{-\cos 2\theta}{4}\Big|_0^{\pi/2}\right)\left(\frac{r^4}{4}\Big|_0^1\right)$$

$$= \frac{-\cos\pi - -\cos 0}{4}\cdot\frac{1}{4} = \frac{2}{4}\cdot\frac{1}{4} = \frac{1}{8} \quad .$$

Ans.: $\frac{1}{8}$ (type=number).

2. Evaluate the iterated integral by converting it to polar coordinates

$$\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy \quad .$$

Sol. This is an iterated integral over a **type-II** region (since the dy integration is outside). Looking at the limits of integration, the region in question is

$$\{(x,y)\,|\,0 \leq y \leq 1\,,\, 0 \leq x \leq \sqrt{1-y^2}\} \quad .$$

This is the quarter-circle, center origin, radius 1, that is located at the first-quadrant. A polar-coordinates description is

$$\{(r,\theta)\,|\,0 \leq \theta \leq \pi/2\,,\, 0 \leq r \leq 1\} \quad .$$

Since $x^2 + y^2 = r^2$ and $dydx = dA = rdrd\theta$, the integral equals:

$$\int_0^{\pi/2} \int_0^1 e^{r^2}\, rdr\, d\theta \quad .$$

This is the set-up. In real life you would go to Maple. But here we have to do it by hand.

The inner integral is:

$$\int_0^1 re^{r^2} dr \quad .$$

Doing the substitution $u = r^2$ we get $du = 2rdr$, and when $r = 0, u = 0$ and when $r = 1, u = 1$. So this equalus

$$\int_0^1 e^u/2 = (1/2)e^u \Big|_0^1 = (1/2)(e^1 - e^0) = \frac{e-1}{2} \quad .$$

Now we do the **outer integral**.

$$\int_0^{\pi/2} \frac{e-1}{2}\, d\theta = \frac{e-1}{2}\theta \Big|_0^{\pi/2} = \frac{e-1}{2} \cdot \frac{\pi}{2} = \frac{(e-1)\pi}{4} \quad .$$

Ans. $\frac{(e-1)\pi}{4}$ (type number).