Solutions to the "QUIZ" for Lecture 14

1. Evaluate the iterated integral

$$\int_0^1 \int_x^{3x} \int_0^y x^2yz\,dz\,dy\,dx \quad .$$

Sol. We first do the **inner** integral

$$\int_0^y x^2yz\,dz = x^2y\int_0^y z\,dz = x^2y\left(\frac{z^2}{2}\Big|_0^y\right) = x^2y\left(\frac{y^2}{2}-\frac{0^2}{2}\right) = \frac{x^2y^3}{2}$$

We next do the **middle** integral

$$\int_x^{3x} \frac{x^2y^3}{2}\,dy = \frac{x^2}{2}\int_x^{3x} y^3\,dy = \frac{x^2}{2}\left(\frac{y^4}{4}\Big|_x^{3x}\right) = \frac{x^2}{2}\left(\frac{(3x)^4-x^4}{4}\right) =$$

$$\frac{x^2}{2}\left(\frac{81x^4-x^4}{4}\right) = \frac{x^2}{2}\left(\frac{80x^4}{4}\right) = 10x^6$$

Finall, we do the **outer** integral

$$\int_0^1 10x^6\,dx = 10\frac{x^7}{7}\Big|_0^1 = \frac{10}{7} \quad .$$

Ans.: $\frac{10}{7}$ (type number).

2. Evaluate the triple integral

$$\int\int\int_E yz\ln(x^5)\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,0\leq x\leq 1\,,\,0\leq y\leq x\,,\,2x\leq z\leq 3x\,\} \quad .$$

Sol. We first **set-up** the **volume integral** as an **iterated integral**, following the region E.

$$\int_0^1 \int_0^x \int_{2x}^{3x} yz\ln(x^5)\;dz\,dy\,dx \quad .$$

We first do the **inner integral**:

$$\int_{2x}^{3x} yz\ln(x^5)\;dz = y\ln(x^5)\int_{2x}^{3x} z\;dz = y\ln(x^5)\frac{z^2}{2}\Big|_{2x}^{3x} =$$

$$y\ln(x^5)\frac{(3x)^2-(2x)^2}{2} = y\ln(x^5)\frac{9x^2-4x^2}{2} = y\ln(x^5)\frac{5x^2}{2} \quad .$$

We next do the **middle** integral:

$$\int_0^x y\ln(x^5)\frac{5x^2}{2}\,dy = \ln(x^5)\frac{5x^2}{2}\int_0^x y\,dy = \ln(x^5)\frac{5x^2}{2}\left(\frac{y^2}{2}\Big|_0^x\right) = \ln(x^5)\frac{5x^2}{2}\cdot\frac{x^2}{2} = \ln(x^5)\frac{5x^4}{4}$$

We finally do the **outer integral**

$$\int_0^1 \ln(x^5)\frac{5x^4}{4}\,dx$$

Doing the **substition** $z = x^5$ we get $dz = 5x^4\,dx$, and when $x = 0, z = 0$, and when $x = 1, z = 1$. So we have

$$\int_0^1 ln(z)\frac{1}{4}\,dz = \frac{1}{4}\int_0^1 ln(z)\,dz \quad .$$

By **integration by parts**

$$\int \ln(z)\,dz = \int_0^1 z' ln(z)\,dz = z\ln z - \int z(ln(z))'\,dz = z\ln z - \int z\cdot(1/z)\,dz = z\ln z - \int dz = z\ln z - z \quad .$$

So we have

$$\frac{1}{4}\int_0^1 ln(z)\,dz = \frac{1}{4}(z\ln z - z)\Big|_0^1 = \frac{1}{4}[(1\ln 1 - 1) - (0\ln 0 - 0)] \quad = \frac{1}{4}[(1\cdot 0 - 1) - (0 - 0)] = -\frac{1}{4}$$

Ans. $-\frac{1}{4}$ (type number).

Comments: 1. Strictly speaking $0\ln 0$ is nonsense, since $\ln 0$ is undefined. What I mean by "$0\ln 0$" is $\lim_{x\to 0^+} x\ln x$ that is well-known to be 0 (or use L'Hôpital in the form $\frac{\ln x}{x^{-1}}$) .

2. I often tell you to **simplify before you integrate**. In this problem, an obvious simplification is $\ln x^5 = 5\ln x$, but in this case it does not make things any simpler (in fact, it makes it slightly more complicated).