Solutions to the "QUIZ" for Lecture 13

Corrected version (Oct. 27, 2020) thanks to Aastha Kasera, who found a typo (and won a dollar!)

1. Change the order of integration in

$$\int_1^4 \int_0^{\ln y} f(x,y)\, dx\, dy \quad .$$

Sol.: The **region** of integration is

$$D = \{(x,y) | 1 \le y \le 4, 0 \le x \le \ln y\}$$

This is a type II (Horizontally Simple) region. The **main road** is along the y-axis from $(0,1)$ to $(0,4)$, and the "side-streets" extend from the y-axis (where $x = 0$) to the curve $x = \ln y$. Drawing it, we get that the projection on the x-axis of that region is between $x = 0$ and $x = \ln 4$, and making this the main road, the vertical side-streets extend from the curve $x = \ln y$, that we should rewrite, from the new perspective as $y = e^x$, to the horizontal line $y = 4$. So a type I description is

$$D = \{(x,y) | 0 \le x \le \ln 4, e^x \le y \le 4\}$$

Finally the integral is

$$\int_0^{\ln 4} \int_{e^x}^4 f(x,y)\, dy\, dx \quad .$$

This is the **ans.** (type: double integral of a general function on a specific domain).

2. Evaluate

$$\int_0^2 \int_{y/2}^1 \frac{1}{(x^2+1)^2}\, dx\, dy \quad ,$$

by inverting the order of integration and evaluating the new iterated integral.

Sol. The region implied by the given iterated integral is

$$D = \{(x,y) | 0 \le y \le 2, y/2 \le x \le 1\}$$

This is a type II description. If you draw it, the **main road** is along the y-axis from $y = 0$ to $y = 2$ and the horizontal side streets are from the line $x = y/2$ to the vertical line $x = 1$. This turns out to be a triangle with vertics $(0,0), (1,0), (1,2)$.

Let's find a type I description of D. The projection on the x-axis is from $x = 0$ to $x = 1$ and the horizontal side streets starts at $y = 0$ and end at the line $x = y/2$ that we now must write at $y = 2x$. So the type I description of D is:

$$D = \{(x,y) | 0 \le x \le 1, 0 \le y \le 2x\} \quad .$$

Now we can **interchange** the order of integration and get

$$\int_0^1 \int_0^{2x} \frac{1}{(x^2+1)^2} \, dy \, dx \quad .$$

We first do the **inner** integral

$$\int_0^{2x} \frac{1}{(x^2+1)^2} \, dy = \frac{1}{(x^2+1)^2} \left(\int_0^{2x} dy \right)$$

$$= \frac{1}{(x^2+1)^2} \left(y \Big|_0^{2x} \right) \quad = \frac{1}{(x^2+1)^2} (2x - 0) \quad = \frac{2x}{(x^2+1)^2} \quad .$$

Now we are ready to do the **outer** integral

$$\int_0^1 \frac{2x}{(x^2+1)^2} \, dx \quad .$$

This calls for a **u-substitution**. $u = x^2 + 1$, $2x \, dx = du$, and when $x = 0$, $u = 1$, and when $x = 1$, $u = 2$.

$$\int_1^2 \frac{1}{u^2} \, du = \int_1^2 u^{-2} \, du = \frac{u^{-1}}{-1} \Big|_1^2$$

$$= \frac{-1}{u} \Big|_1^2 = \frac{-1}{2} - \frac{-1}{1} = \frac{1}{2} \quad .$$

Ans.: $\frac{1}{2}$ (type: number).