Solutions to the "QUIZ" for Lecture 23

1. Determine whether or not the vector field is conservative. If it is, find a function f such that $\mathbf{F} = \bigtriangledown f$.

$$\mathbf{F}(x, y, z) = (3x^2y^3z^3 + yz)\,\mathbf{i} + (3x^3y^2z^3 + xz)\,\mathbf{j} + (3x^3y^3z^2 + xy)\,\mathbf{k}$$

Sol.: Here:

$P = 3x^2y^3z^3 + yz$, $Q = 3x^3y^2z^3 + xz$, $R = 3x^3y^3z^2 + xy$.

We have:

$\frac{\partial P}{\partial y} = 9x^2y^2z^3 + z \quad , \quad \frac{\partial Q}{\partial x} = 9x^2y^2z^3 + z \quad ,$

these are equal.

$\frac{\partial P}{\partial z} = 9x^2y^3z^2 + y \quad , \quad \frac{\partial R}{\partial x} = 9x^2y^3z^2 + y \quad ,$

these are equal.

$\frac{\partial Q}{\partial z} = 9x^3y^2z^2 + x \quad , \quad \frac{\partial R}{\partial y} = 9x^3y^2z^2 + x \quad ,$

these are equal.

So $\mathbf{F}$ is indeed conservative.

We now need to find a function $f(x, y, z)$ such that

$$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle P, Q, R \rangle \quad .$$

Since $\frac{\partial f}{\partial x} = P$,

$$f(x, y, z) = \int (3x^2y^3z^3 + yz)\, dx = x^3y^3z^3 + xyz + g(y, z) \quad ,$$

where $g(y, z)$ is yet-to-be-determined. Using $\frac{\partial f}{\partial y} = Q$,

$$3x^3y^2z^3 + xz + \frac{\partial g}{\partial y} = 3x^3y^2z^3 + xz \quad ,$$

giving

$$\frac{\partial g}{\partial y} = 0 \quad ,$$

So

$$g(y, z) = \int 0\, dy = 0 + h(z) \quad ,$$

where $h(z)$ is yet-to-be-determined. Going back above:

$$f(x, y, z) = x^3y^3z^3 + xyz + h(z) \quad ,$$

Using $\frac{\partial f}{\partial z} = R$, we get:

$$3x^3y^3z^2 + xy + h'(z) = 3x^3y^3z^2 + xy \quad ,$$

giving $h'(z) = 0$ so $h(z) = C$ and we have

$$f(x,y,z) = x^3y^3z^3 + xyz + C \quad ,$$

but we can forget about the C (we were asked to find *a* potential function, not all of them), so

Ans. to second part: $f(x,y,z) = x^3y^3z^3 + xyz$.

2. Evalute

$$\int_C 5y\,dx + 10x\,dy \quad ,$$

where C is the closed curve consisting of the boundary of the rectangle

$$\{\,(x,y)\,|\,0 \le x \le 1 \quad , \quad 0 \le y \le 1\,\}.$$

Sol.: We are supposed to use **Green's Theorem**.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad .$$

Since no orientation is mentioned we take the **default** one of *positive* (counter-clockwise).

In this problem $P = 5y$ and $Q = 10x$, so we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10 - 5 = 5 \quad ,$$

and the vector-field line-integral that we have to compute equals:

$$\int\int_D 5\,dA = 5\int\int_D 1\ dA = 5Area(D) \quad ,$$

(since an **area integral** with the integrand being 1 equals the **area**). By elementary-school geometry, the area of D is $1 \cdot 1 = 1$, so we have:

Ans.: 5 (type number).