Solutions to the 'QUIZ" for Lecture 20

Version of Nov. 22, 2020, thanks to Shubin Xie (who won 5 dollars)

1. Find an equation for the tangent plane to the parametric surface

$$x = v^2 \quad , \quad y = u + v \quad , \quad z = u^2 \quad ,$$

at the point $(1, 2, 1)$. Simplify as much as you can!

Sol. Here

$$\mathbf{r}(t) = \langle v^2, u + v, u^2 \rangle$$

Taking derivatives with respect to u and v, we get

$$\mathbf{r}_u = \langle 0, 1, 2u \rangle \quad ,$$

$$\mathbf{r}_v = \langle 2v, 1, 0 \rangle \quad .$$

Next, we have to find out what are u and v **at** the point $(1, 2, 1)$. We have to solve, for u, v:

$$1 = v^2, 2 = u + v, 1 = u^2$$

From the first equation $v = -1$ or $v = 1$, from the last, $u = -1$ or $u = 1$, but to satisfy the second equation, only $u = 1$ and $v = 1$ are OK. So we know that at the designated point, $u = 1, v = 1$.

Plugging these above gives:

$$\mathbf{r}_u = \langle 0, 1, 2 \rangle \quad ,$$

$$\mathbf{r}_v = \langle 2, 1, 0 \rangle \quad .$$

To find the normal, we take the cross-product

$$\mathbf{n} = \langle 0, 1, 2 \rangle \times \langle 2, 1, 0 \rangle = \langle -2, 4, -2 \rangle \quad .$$

(you do it!).

The equation of the tangent plane is

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \mathbf{n} = 0 \quad ,$$

So, in this problem, it is

$$\langle x - 1, y - 2, z - 1 \rangle \cdot \langle -2, 4, -2 \rangle = 0 \quad ,$$

that spells out to:

$$(-2)(x - 1) + 4(y - 2) + (-2)(z - 1) = 0 \quad .$$

Dividing both sides by -2 and simplifying, we get

$$x - 2y + z = -2 \quad .$$

Ans. $x - 2y + z = -2$ (type: Eq. of a plane).

2. Evaluate the surface integral

$$\int\int_S z\, dS \quad ,$$

where S is the triangular region with vertices $(2,0,0),(0,2,0),(0,0,2)$.

Sol. We first find the equation of the plane passing through the three points. This turns out to be

$$x + y + z = 2 \quad .$$

(in this easy case you can do it by "inspection" (adding up the three coordinates always gives you 2, in general you would have to work hard, doing $\mathbf{n} = \mathbf{AB} \times \mathbf{AC}$ etc.)

Expressing this plane in **explicit** form, we have

$$z = 2 - x - y \quad .$$

The relevant formula is:

$$\int\int_S f(x,y,z)dS = \int\int_D f(x,y,g(x,y))\sqrt{1 + g_x^2 + g_y^2}\, dx\, dy \quad ,$$

where D is the projection of the region on the xy-plane.

Here $g(x,y) = 2 - x - y$, so $g_x = -1, g_y = -1$, and $\sqrt{1 + g_x^2 + g_y^2} = \sqrt{3}$. So

$$\int\int_S z\, dS = \int\int_D (2 - x - y)\,\sqrt{3} \quad .$$

It still remains to find out the region D. The plane $z = 2 - x - y$ meets the xy plane (alias $z = 0$) at the line $x + y = 2$. Since $x \geq 0, y \geq 2$ the region D is

$$D = \{(x,y) | x \geq 0, y \geq 0, x + y \leq 2\} \quad .$$

A type I description is

$$D = \{(x,y) | 0 \leq x \leq 2, 0 \leq y \leq 2 - x\} \quad .$$

So we get

$$\int_0^2 \int_0^{2-x} (2 - x - y)\,\sqrt{3}\, dy\, dx \quad .$$

The inner integral is

$$\sqrt{3}\int_0^{2-x} (2 - x - y)\, dy = \sqrt{3}((2-x)\,y \; - \; \frac{y^2}{2}\Big|_0^{2-x} = \frac{\sqrt{3}}{2}(2-x)^2 \quad .$$

The outer integral is:

$$\frac{\sqrt{3}}{2}\int_0^2 (2-x)^2\, dx = -\frac{\sqrt{3}}{2}\frac{(2-x)^3}{3}\Big|_0^2 = \frac{4}{3}\sqrt{3} \quad .$$

Ans.: $\frac{4}{3}\sqrt{3}$ (type: number).