**Dr. Z's Math251 LAST Handout [The Meaning of It All]**

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**Concept 1: Indefinite Integral**

**Notation**: $\int f(x)\,dx$ .

**Input**: A *function* $f(x)$ of the single-variable $x$.

**Output**: A **function** $F(x)$ .

**Meaning:** $F(x)$ is a function such that $F'(x) = f(x)$. Since there are many such functions (for example if $f(x) = x$, then $F(x) = x^2/2$, $F(x) = x^2/2 + 1$, $F(x) = x^2/2 + 11$, etc. all have this property, we write $+C$ for *arbitrary* constant, to indicate that this is really a *family of answers*.

**How to compute it:** Memorize the answer for basic functions and use **integration techniques** for complicated ones. Better still: use Maple (but not on exams!).

**Concept 2: Definite Integral**

**Notation**: $\int_a^b f(x)\,dx$ .

**Input**: A *function* $f(x)$ of the single-variable $x$, and two *numbers* $a$ and $b$.

**Output**: A **number**.

**How to Compute it**: Find the *indefinite* integral $\int f(x)\,dx$, let's call it $F(x)$, and do $F(b)-F(a)$.

**Meanings**: 1. If $f(x)$ is positive for $a \le x \le b$, then it is the area under the curve $y = f(x)$, above the $x$ axis, and between the vertical lines $x = a$ and $x = b$.

2. If you have a one-dimensional wire stretched out from $x = a$ to $x = b$, and its (linear) density-function is $f(x)$, then this is its **mass**.

**Important Special case:** when $f(x) = 1$, $\int_a^b dx = b - a$, the length of the interval $b - a$.

**Concept 3: Line-Integral of the arclength kind**

**Notation**: $\int_C f(x, y, z)ds$ .

**Inputs**: a function of **three** variables $(x, y, z)$ (or sometimes **two** $(x, y)$)) and a **curve** described (usually) in **parametric** representation. $x = Expression_1(t), y = Expression_2(t), z = Expression_3(t)$ $(a \le t \le b)$ or equivalently in **vector notation** $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ $(a \le t \le b)$.

**Output**: a **number**.

**How to Compute it**: replace $ds$ by $|\mathbf{r}'(t)|dt$, $\int_C$ by $\int_a^b$ and $f(x,y,z)$ by $f(x(t),y(t),z(t)$, getting a definite integral in $t$.

**Meanings**: 1. If you bulid your two-dimensional house along the curve, with the floor being the curve and the ceiling-height given by the function $f(x,y,z)$, it is the area of that house.

2. If it is a wire whose linear density is given by $f(x,y,z)$ then it is its mass .

**Important Special case:** If $f(x,y,z)=1$ we get the **length** of the curve (also called **arclength**).

**Concept 3': Line-Integral of the Vector-Field kind**

**Notation**: $\int_C \mathbf{F}\cdot d\mathbf{r}$

**Inputs**: a **vector-field** of **three** variables $(x,y,z)$, $\mathbf{F}=\langle P,Q,R\rangle$, (or sometimes **two** $(x,y)$ and $\mathbf{F}=\langle P,Q\rangle$) and a **curve** described (usually) in **parametric** representation. $x=Expression_1(t), y=Expression_2(t), z=Expression_3(t)$ $(a\le t\le b)$ or equivalently in **vector notation** $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$ $(a\le t\le b)$.

**Output**: a **number**.

**How to Compute it**: replace $d\mathbf{r}$ by $\mathbf{r}'(t)dt$, $\int_C$ by $\int_a^b$ and $\mathbf{F}$ by what you get by plugging-in in $P,Q,R$ the expressions in terms of $t$ for $x,y,z$. Then compute the dot-product $\mathbf{F}\cdot\mathbf{r}'(t)dt$ and integrate from $t=a$ to $t=b$. Equivalently do $\int Pdx+Qdy+Rdz$ with $dx=x'(t)dt$, $dy=y'(t)dt$, $dz=z'(t)dt$.

**Meanings**: Several in physics. Most important, the **work** done by a **force F** moving along $C$.

**Important Special case:** None.

**Concept 4: Iterated Integral of two variables**

**Notation**: $\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ (type I)

or

$\int_a^b\int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$ (type II)

**Inputs**: a **function** $f(x,y)$ of two variables, **numbers** $a$ and $b$ and **functions** (that sometimes happen to be constant functions hence may be numbers) $g_1(x),g_2(x)$ (for type I) or $g_1(y),g_2(y)$ (for type II).

**Output**: a **number**.

**Watch out for nonsense:** $\int_x^{2x}\int_0^1 f(x,y)dydx$ is utter-nonsense (**outside** limits-of-integration can only be numbers).

**How to Compute it**: First you do the **inner-integral**, getting a function of the outside variable-of-integration, then you do the **outside-integral** getting a number.

**Meanings**: these come up when evaluating double-integrals (alias area-integrals).

**Important Special case:** None.

**Concept 5: Iterated Integral of three variables**

Analogous to two variables, but now you have an **inside** integral, a **middle** integral, and an **outside** integral.

**Concept 6: Double Integral (alias Area-integral)**

**Notation**: $\int \int_D f(x, y)\, dA$ .

**Inputs**: a **function** $f(x, y)$ of two variables, and a **region** $D$ in the plane ($xy$-plane).

**Output**: a **number**.

**How to Compute it**: Express the region $D$ either in type-I style, or type-II style, or polar. Then convert the double-integral into an **iterated integral**.

**Meanings**: 1. If $f(x, y)$ is positive, and you build a building whose floor is the region $D$, and whose ceiling-height above a typical point $(x, y)$ is given by the function $f(x, y)$, then it is the **volume** of that building.

2. If you have a plate whose shape is $D$ and $f(x, y)$ is its (surface) **density function**, then it is the **mass** of the plate.

**Important Special case:** if $f(x, y)$ is 1 then you get the **area** of $D$.

**Concept 7: Surface Integral of the scalar kind**

**Notation**: $\int \int_S f(x, y, z)\, dS$

**Inputs**: a **function** $f(x, y, z)$ of three variables, and a **surface** $S$ (either given **explicitly** as $z = g(x, y)$ or **parametrically** as as $\mathbf{r}(u, v)$.

**Output**: a **number**.

**How to Compute it**: In the explicit case: find the **projection** on the $xy$-plane, call it $D$, then replace $dS$ by $\sqrt{1 + (g_x)^2 + (g_y)^2} dA$ and do the resulting double-integral (not forgetting to replace $z$ by $g(x, y)$ so you won't get any mention of $z$.)

In the parametric case: Let $D$ be the **parameter-space** region (usually given by the problem). Replace $x, y, z$ by their expression in $(u, v)$ and replace $dS$ by $|r_u \times r_v| dA$, and do the resulting double-integral.

**Meanings**: 1. If you have a building whose floor is the surface $S$, and whose ceiling-height above a typical point $(x, y, z)$ is given by the function $f(x, y, z)$, then it is the **volume** of that building.

2. If you have a piece of material that fits on S and $f(x, y, z)$ is the density function, then it is the **mass**.

**Important Special case:** if $f(x, y, z)$ is 1 then you get the **surface-area** of $S$.

**Concept 7': Surface Integral of the vector-field kind**

**Notation**: $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ .

**Inputs**: a **vector-field** $\mathbf{F}(x, y, z)$ of three variables, and three components (called $P, Q, R$) and a **surface** $S$ (either given **explicitly** as $z = g(x, y)$ or **parametrically** as as $\mathbf{r}(u, v)$ ) .

**Output**: a **number**.

**How to Compute it**: In the explicit case, $z = g(x, y)$ find the projection of $S$ on the $xy$-plane, call it $D$, and find the double-integral

$$\int\int_D \left( -P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R \right) dA$$

In the parametric case: Let $D$ be the **parameter-space** region (usually given by the problem). replace $x, y, z$ by their expression in $(u, v)$ and replace $d\mathbf{S}$ by $(r_u \times r_v)dA$, take the dot-product and do the resulting double-integral.

**Meanings**: Several in physics (most notably Electricity and Magnetism).

**Important Special case:** None.

**Concept 8.: Triple Integral (alias Volume-Integral)**

**Notation**: $\int\int\int_E f(x, y, z)\, dV$ .

**Inputs**: a **function** $f(x, y, z)$ of three variables, and a **region** $E$ in space.

**Output**: a **number**.

**How to Compute it**: Express the region $E$ either in $x - y - z$ order or $x - z - y$ order etc., or in cylindrical, or in spherical coordinaters. Then convert the triple-integral into an **iterated integral** and do it **one-step-at-a-time** from inside to outside.

**Meanings**: 1. If $f(x, y, z)$ is positive, and you build a **four-dimensional** building whose "floor" is the region $E$, and whose ceiling-height above a typical point $(x, y, z)$ is given by the function $f(x, y, z)$, then it is the **four-dimensional "volume"** of that building.

2. If you have a solid body whose shape is $E$ and $f(x,y,z)$ is its **density function**, then it is the **mass** of that solid-body.

**Important Special case:** if $f(x,y,z)$ is 1 then you get the **volume** of $E$.

**Concept 9: Gradient**

**Notation**: $\bigtriangledown f(x,y,z)$ .

**Inputs**: a **function** $f(x,y,z)$ of three variables .

**Output**: a **vector-field**.

**How to Compute it**: $\bigtriangledown f = \langle f_x, f_y, f_z \rangle$ .

**Meaning**: the **magnitude** of $\bigtriangledown f$ is the **maximum rate of change** and its direction is the **direction where it changes the fastest**.

**Concept 9: Curl**

**Notation**: $curl\,\mathbf{F}$ (in math) or $\bigtriangledown \times \mathbf{F}$ (in physics)

**Input**: a **vector-field** $\mathbf{F}(x,y,z)$ of three components and three variables, $\langle P, Q, R \rangle$.

**Output**: another **vector-field**.

**How to Compute it**:

$$\bigtriangledown \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad .$$

**Meaning**: very important in Electricity and Magnetism.

**Concept 10: Divergence**

**Notation**: $div\mathbf{F}$ (in math) or $\bigtriangledown \cdot \mathbf{F}$ (in physics) .

**Input**: a **vector-field** $\mathbf{F}(x,y,z)$ of three components and three variables, $\langle P, Q, R \rangle$.

**Output**: a **function** of $(x,y,z)$.

**How to Compute it**: $P_x + Q_y + R_z$.

**Meaning**: very important in Electricity and Magnetism.

**Important relationships between concepts**

**Green's Theorem** relates the notion of **line-integral** (of the vector-field kind over a **closed**

curve) in **two** dimensions with the notion of **area-integral** (a.k.a. as **double integral**).

**Stokes's Theorem** relates the notion of **line-integral** (of the vector-field kind) in **three** dimensions with the notion of **surface-integral** (of the vector-field kind) over an **open** surface [one that has a bounding-curve].

**The Divergence Theorem** relates the notion of **surface-integral** (of the vector-field kind) over a **closed** surface with the notion of **triple-integral** (a.k.a. volume-integral) .

**Important Simplifications**

1. For **any** function $f(x, y, z)$ (that is "nice") $curl\,(\bigtriangledown f) = \mathbf{0}$.

More explicitly: If you take any function $f(x, y, z)$ (you name it!) and first take its **gradient** getting some vector field (a vector of functions). Then take the curl of that vector-field, and **surprise** you get the **0** vector-field.

2. For **any** vector-field $\mathbf{F}(x, y, z)$ (that is "nice") $div\,(curl\,\mathbf{F}) = 0$.

More explicitly: If you take any vector-field $\mathbf{F}(x, y, z)$ (you name it!) and first take its **curl** getting some other vector field, then take the **divergence** of that vector-field, and **surprise** you get the 0-function.

How to use these fact? Suppose that I give you an extremely complicated function like

$$f(x, y, z) = \cos(e^{\cos xyz + \sin z} + x + y + z^3)(x^3 + 7xyz - y^6)$$

and ask you to compute $curl(\bigtriangledown f)$. If you do it stupidly, you would first find $\bigtriangledown f$, and then take the curl of that. This would take you two hours and you won't have time to do anything else. But if you **knew** the fact that $curl\,(\bigtriangledown f)$ is **always 0** it would take you one second.

**Important shortcuts**

1. If you have an area-integral with the integrand being a plain number of a region whose area you know from middle-school, for example

$$\int\int_D 5\,dA \quad ,$$

You first take the 5 in front

$$5\int\int_D dA \quad ,$$

and use the fact that when the integrand is 1 the **area-integral** becomes plain **area** so the answer is 5 times the area of $D$.

Remember: the area of a rectangle is: length times width;

the area of a triangle is: base times height over 2.

The area of a circle radius $r$ is $\pi r^2$.

The area of a semi-circle radius $r$ is $(\pi r^2)/2$.

2. Similarly if you have a **surface-integral** with the integrand being a number (remember the surface area of a sphere radius $R$ is $4\pi R^2$, that of an open hemisphere half of that, that of a closed cylinder radius $R$ height $h$ is $2\pi Rh + 2\pi r^2$) .

3. Similarly if you have a **volume-integral** with the integrand being a number (remember the volume of a sphere radius $R$ is $(4\pi/3)R^3$, that of a hemisphere half of that, that of a cylinder radius $R$ height $h$ is $\pi R^2 h$) .

4. Similarly if you have a **line-integral** (of the $ds$ kind) with the integrand being a number (remember the length (i.e. circumference) of a circle radius $R$ is $2\pi R$).

5. In **polar** and **cylindrical** coordinates the "phrase" $x^2 + y^2$ is $r^2$.

6. In **spherical** coordinates the "phrase" $x^2 + y^2 + z^2$ is $\rho^2$.

**Use and Abuse of Nonsense Detection**

length, area, and volume can **never** be negative numbers but **line-integrals**, **area-integrals** and **volume-integrals** often are.