Solutions and Commentary on MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Exam 2, Monday, Nov. 23, 2020, 8:40-10:40am

Types: Number, Function of *variable*(s), 2D vector of numbers, 3D vector of numbers, 2D vector of functions, 3D vector of functions, equation of a plane, parametric equation of a line, equation of a line, equation of a surface, equation of a line, DNE (does not exist), parametric equation of surface, double integral of an abstract function.

---

1. (10 pts.) Compute the line integral

$$\int_C yz\,dx \,+\, (xz+z)\,dy \,+\, (xy+y+1)\,dz \quad ,$$

over the path

$$\mathbf{r}(t) = \langle e^{t^3}, t^2e^{t^4}, te^{t^7}\rangle \quad , \quad 0 \le t \le 1 \quad .$$

Explain!

The **type** of the answers is: Number.

---

ans. $e + e^2 + e^3$

Comment: This was meant to be done the **clever way**. People who did it directly, and then used Maple to evaluate it, and got the right answer, still got full credit, but in the Final I may specify "do not use Maple" (or any other software). Also I gave full credit for the answer in decimals. The perfect answer should be in terms of e. In the final exam I will insist on it (but will mention it explicitly).

Sol. Whenever the vector field is **conservative**, then it is much easier to use the **Fundamental Theorem of Line Integrals**. Taking the $curl(\mathbf{F})$ gives $< 0,0,0 >$ (you do it!) hence there is a **potential function** $f(x,y,z)$ such that $grad(f) =< yz, xz{+}z, xy{+}y{+}1 >$. i.e

$$f_x \,=\, yz \quad , \quad f_y \,=\, xz+z \quad , \quad f_z \,=\, xy+y+1 \quad .$$

From $f_x = yz$ we get $f(x,y,z) = xyz + \phi(y,z)$ where $\phi(y,z)$ is **to be determined**. Using $f_y = xz+z$, we get $xz + \phi_y(y,z) = xz+z$, hence $\phi_y(y,z) = z$. Integrating with respect to y, we get $\phi(y,z) = yz + \psi(z)$ where $\psi(z)$ is **to be determined**. We currently have $f(x,y,z) = xyz + yz + \psi(z)$. Using $f_z = xy + y + 1$, we get $xy + y + \psi'(z) = xy + y + 1$ so $\psi'(z) = 1$ and so $\psi(z) = z + C$ (but C is not important for this problem) so

Crucial First Step: The potential function is $f(x,y,z) = xyz + yz + z$.

Comment: At this step it is a good idea to **check** that $grad(f)$ is indeed $< yz, xz + z, xy + y + 1 >$. Do it!

Now we almost done. By the **Fundamental theorem of Line Integrals** the value is $f(End) - f(Start)$.

The starting point is $\mathbf{r}(0) = (1,0,0)$, the ending point is $\mathbf{r}(1) = (e,e,e)$, so the final answer is

$$f(e,e,e) - f(1,0,0) = e^3 + e^2 + e - 0 = e^3 + e^2 + e \quad .$$

2. (10 points) By changing the order of integration, if necessary, evaluate the double-integral

$$\int_0^5 \int_{(y/5)^{1/3}}^{1} \sin x^4 \, dx \, dy$$

The **type** of the answer is: Number

---

ans. $\frac{5}{4}(1 - \cos 1)$

Sol. The **region of integration** is, in **Type II format**

$$D = \{(x,y) : 0 < y < 5 \quad , \quad (y/5)^{1/3} < x < 1\} \quad .$$

Plotting it, and realizing that the curve $x = (y/5)^{1/3}$ can be written in the usual format as $y = 5x^3$ (solve for y in terms of x), the same region in **Type I format** is

$$D = \{(x,y) \, : \, 0 < x < 1 \quad , \quad 0 < y < 5x^3\} \quad .$$

This leads to the **iterated integral**

$$\int_0^1 \int_0^{5x^3} \sin\, x^4 \, dy \, dx \quad .$$

The **inner integral** is

$$\int_0^{5x^3} \sin\, x^4 \, dy \;=\; \sin\, x^4 \int_0^{5x^3} \, dy \;=$$

$$= \sin\, x^4 (y|_0^{5x^3}) \;=\; 5x^3 \sin\, x^4 \quad .$$

The **outer integral** is

$$\int_0^1 5x^3 \sin\, x^4 \, dx \quad .$$

Doing the substitution $u = x^4$ we get that this

$$\frac{5}{4}\int_0^1 \sin u \, du = \frac{5}{4}(-\cos u)\Big|_0^1 = \frac{5}{4}(-\cos 1 - (-\cos 0)) = \frac{5}{4}(1 - \cos 1) \quad .$$

Comment: Once again, I forgot to state "do not use Maple". People who did the clever change of order, and then used Maple got full credit. In the Final I may add the restrictions, "do not use a computer" and "explain everything".

3. (10 points) Find the equation of the tangent plane at the point $(1, 1, 1)$ to the surface given parametrically by

$$x(u, v) = u^3 v \quad , \quad y(u, x) = uv \quad , \quad z(u, v) = uv^3 \quad , \quad -\infty < u < \infty \quad , \quad -\infty < v < \infty \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

The **type** of the answer is: Equation of a Plane

---

ans. $z = -x + 4y - 2$

Sol. The parametric equation of the surface in **vector notation** is

$$\mathbf{r}(u, v) = \langle u^3 v \, , \, uv \, , \, uv^3 \rangle \quad .$$

We have

$$\mathbf{r}_u = \langle 3u^2 v \, , \, v \, , \, v^3 \rangle \quad .$$

$$\mathbf{r}_v = \langle u^3 \, , \, u \, , \, 3uv^2 \rangle \quad .$$

Now it is time to **plug it in**. But first we need to find the values of u and v that correspond to $(1,1,1)$. This can be easily seen to be $u = 1, v = 1$.

$$\mathbf{r}_u(1,1) = \langle 3 \, , \, 1 \, , \, 1 \rangle \quad .$$

$$\mathbf{r}_v(1,1) = \langle 1 \, , \, 1 \, , \, 3 \rangle \quad .$$

In order to find a **Normal vector**, we find the **cross product** $\mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{N} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$= \mathbf{i}(1 \cdot 3 - 1 \cdot 1) - \mathbf{j}(3 \cdot 3 - 1 \cdot 1) + \mathbf{k}(1 \cdot 3 - 1 \cdot 1) = 2\mathbf{i} - 8\mathbf{j} + 2\mathbf{k} = \langle 2, -8, 2 \rangle \quad .$$

The point, (x_0, y_0, z_0) is $(1,1,1)$, so using

$$\mathbf{N}.\langle x - x_0, y - y_0, z - z_0 \rangle \; = \; 0 \quad ,$$

we get

$$\langle 2, -8, 2 \rangle . \langle x - 1, y - 1, z - 1 \rangle = 0 \quad .$$

Dividing by 2 this gives

$$(x-1) - 4(y-1) + (z-1) = 0 \quad .$$

Simplifying, and solving for z gives $z = -x + 4y - 2$.

4. (10 points) Let $f(x,y,z) = e^{\cos x^2 + \sin xyz + \cos xz}$, and let

$$\mathbf{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

Let C be the curve

$$r(t) = \langle \cos t, t, \sin t \rangle \quad , \quad 0 \le t \le 2\pi \quad .$$

Find the value of the line-integral

$$\int_C \mathbf{F}.d\mathbf{r} \quad .$$

Explain! Just giving the answer will give you no credit.

The **type** of the answer is: Number

---

ans. 0

Comment: The original version claimed that C is closed. I wrote email to everyone telling them to delete the word "closed". It so happens that the answer is the same either way, 0, and I gave full credit to both versions.

Sol. This obviously calls for the **Fundamental Theorem of Line Integrals**. **F** is $grad(f)$ so the answer is $f(End) - f(Start)$. The starting point is $\langle 1, 0, 0 \rangle$, and the end point is $\langle 1, 2\pi, 0 \rangle$. So the answer is

$$e^{\cos 1 + 1} - e^{\cos 1 + 1} = 0.$$

5. (10 points) Evaluate the triple integral

$$\int_R (x^2 + y^2 + z^2)^3 \, dx \, dy \, dz \quad ,$$

where R is the region in 3D space given by

$$\{(x, y, z) \,|\, x^2 + y^2 + z^2 \le 1 \quad , \quad x, y, z \ge 0\} \quad .$$

The **type** of the answer is: Number

---

ans. $\frac{\pi}{18}$

Sol. This calls for **spherical coordinates**. Recall that $dx\,dy\,dz = \rho^2 \sin\phi\, d\rho d\phi d\theta$. So the **integrand** is (recall that $x^2 + y^2 + z^2 = \rho^2$) $(\rho^2)^3 \rho^2 \sin\phi = \rho^8 \sin\phi$. We also need the region in these coordinates. It is

$$\{(\rho, \phi, \theta) : \rho < 1\,,\, 0 < \phi < \frac{\pi}{2}\,,\, 0 < \theta < \frac{\pi}{2}\} \quad .$$

Hence the required integral is

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^8 \sin\phi\, d\rho\, d\phi\, d\theta \quad .$$

This equals

$$\left(\int_0^1 \rho^8\, d\rho\right) \left(\int_0^{\pi/2} \sin\phi\, d\phi\right) \left(\int_0^{\pi/2} d\theta\right) = \frac{1}{9} \cdot 1 \cdot \frac{\pi}{2} = \frac{\pi}{18} \quad .$$

VERY FREQUENT ERROR: Many people did not see the $x, y, z > 0$ and integrated over the **whole** sphere, with limits of integration in ϕ from 0 to π and in θ from 0 to 2π. **READ THE QUESTION!**. I gave lots of partial credit for people who made this error, but I am disappointed that so many people did not notice it. I gave you half an hour to check and double-check! I am willing to bet that many people used the whole two hours to do problems, and checked very quickly, or not at all.

Warning: A similar error (not reading the question properly and doing a different problem), in the final exam will give you **zero points**.

Comment: If you are **really advanced**, you could have said that since the integrand is symmetric with respect to exchanging x and $-x$, exchanging y and $-y$, and exchanging z and $-z$, the desired value is one-eighth of the answer of integrating over the whole sphere. So if you prefer to integrate over the whole sphere, you could, get, $\frac{4\pi}{9}$, but then you have to explain (using symmetry) why you need to divide by 8. This argument is 'dangerous', since it is only applicable for symmetric integrands.

6. (10 points) Evaluate the double integral

$$\int_{-3}^0 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^2\, dy\, dx$$

The **type** of the answer is: Number

ans. $\frac{243}{4}$

Sol. The region, in type I notation (using the usual, rectangular, coordinates) is

$$\{(x,y) : -3 < x < 0, 0 < y < \sqrt{9-x^2}\}$$

A little diagram shows that it is the **quarter circle** center origin, radius 3 that lies to the left of y-axis and above the x-axis, in other words, the quarter circle of radius 3 that lies in the **second quadrant**.

That diagram shows that the region, expressed in the polar language is

$$\{(r,\theta) : 0 < r < 3, \frac{\pi}{2} < \theta < \pi\} \quad .$$

Converting to polar coordinates we have that the integrand is $r^4 \cdot r = r^5$.

So we have to compute

$$\int_0^3 \int_{\pi/2}^{\pi} r^5 \, d\theta \, dr \; = \; (\int_0^3 r^5 \, dr) \cdot (\int_{\pi/2}^{\pi} d\theta) \; = \; (\frac{r^6}{6}|_0^3) \cdot (\pi - \frac{\pi}{2}) =$$

$$\frac{3^6 \pi}{6 \cdot 2} = \frac{243}{4}\pi \quad .$$

Comment: Maple is smart enough to do it directly. Since I allowed Maple, and did not say not to use it, I gave full credit to people who got the right answer using Maple. If such a problem will come up in the Final Exam, I would probably say "...by converting to polar coordinates and doing it completely by hand..."

7. (10 points altogether) Decide whether the following limits exist. If it does find them. If it does not **Explain** why not?

(a) $(2\ points)$ $\lim_{(x,y)\to(\pi/2,\pi/2)} \frac{\cos x + \sin x}{x+y}$, (b) $(2\ points)$ $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y}$,

(c) $(2\ points)$ $\lim_{(x,y)\to(0,0)} \dfrac{x-y}{x^2-y^2}$, (d) $(4\ points)$ $\lim_{(x,y)\to(1,1)} \dfrac{x+y-2}{2x+y-3}$,

---

ans. (a) $\frac{1}{\pi}$ (b) 0 (c) DNE (d) DNE

Sol. (a) Just plug-it-in. There are no issues and the answer is $\frac{1}{\pi}$

(b): Now we get 0/0, but simplifying we get $x+y$. Now there is no issue in plugging-it-in and we get 0.

(c) Now we simplify and get $1/(x+y)$. Now we plug-it-in, and get 1/0, so obviously DNE.

(d) This is the **hardest**, and it is similar to a problem in a previous exam. I am very disappointed that quite a few peple did not get it. Once again plugging-it-in gives 0/0. Simplifying fails, so we explore the function from various direction, and see the limits. If you use the line $(x,y) = (1+t, 1)$ parallel to the x axis, and let t go to 0 you would get $\frac{1}{2}$ so the limit in the sense of calc1 is $\frac{1}{2}$. If you use the line $(x,y) = (1, 1+t)$, and let t go to 0 you would get 1. Since there is **no agreement**, the limit (in the sense of calc3) **does not exist**).

Comment: A more general argument would be to consider the line through $(1,1)$ with slope m $y = 1 + m(x-1)$ and get that the limit as m goes to 0 is $\frac{m+1}{m+2}$. Since it depends on the slope there is no agreement.

8. (10 points) Compute the line integral $\int_C f\, ds$ where

$$f(x,y,z) = xyz$$

and C is the line segment starting at $(0,0,0)$ and ending at $(1,2,-3)$

The **type** of the answer(s) is: Number

---

ans. $-\frac{3}{2}\sqrt{14}$.

Sol.: We first need a parametric representation $\mathbf{r}(t)$. Recall that s **line segment** from P to Q is $P + t(Q - P)$, where t goes from 0 to 1. In this case it is

$$\mathbf{r}(t) = \langle t, 2t, -3t \rangle \quad .$$

We have

$$\mathbf{r}'(t) = \langle 1, 2, -3 \rangle \quad .$$

So

$$ds = |\mathbf{r}'(t)|\, dt = \sqrt{1^2 + 2^2 + (-3)^2}\, dt \quad .$$

Replacing x, y and z in $f(x,y,z)$ by t, $2t$, $-3t$, respectively, gives $(t)(2t)(-3t) = -6\,t^3$, so the desired integral is

$$\int_0^1 -6t^3\sqrt{14}\, dt \;=\; -\sqrt{14}\frac{6t^4}{4}|_0^1 \;=\; -\frac{3}{2}\sqrt{14} \quad .$$

9. (10 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if $\mathbf{F}$ is

$$\mathbf{F} \;=\; \langle z, z, x \rangle \quad ,$$

and S is the oriented surface

$$z = 9 - x^2 - y^2 \quad , x \geq 0, y \geq 0, z \geq 0$$

with **downward pointing** normal.

The **type** of the answer is Number

---

ans. $-\frac{693}{5}$.

Sol.: In this class we are only supposed to know how to compute **vector-field surface integrals** over surfaces that are given in **explicit** format $z = g(x, y)$ for some function of x and y. The more general case of surfaces defined parametrically is complicated, and can be found in the book, and this problem can be done that way using the parametrization $\langle x, y, 9 - x^2 - y^2 \rangle$ and thinking of x and y as parameters. But **not** recommended.

The good way of doing it is using the formula given in p.4 of handout 16.5

$$\int\int_\Omega \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad ,$$

where Ω is the "floor-plan". Sometimes it is given directly (like in the last problem of handout 16.5), but sometimes, like in this problem, you have to figure it out yourself.

The way to find Ω is to set $g(x, y) = 0$, so in this problem the region is the inside of the circle $x^2 + y^2 = 9$, i.e. the circle of radius 3 and center the origin. **But** note the $x, y > 0$, so the relevant 'floor-plan' is the quarter disc in the first quadrant. Here we have

$$P = z \quad , \quad Q = z \quad , \quad R = x \quad ,$$

$$\frac{\partial g}{\partial x} = -2x \quad , \quad \frac{\partial g}{\partial y} = -2y \quad .$$

So what we need (with the usual upward direction) is

$$\int\int_\Omega (-(z)(-2x) - z(-2y) + x)dA = \int\int_\Omega (2x + 2y)z + x)dA \quad ,$$

But don't forget to replace z by $9 - x^2 - y^2$, getting

$$\int\int_\Omega (2x + 2y)(9 - x^2 - y^2) + x)dA \quad .$$

and in usual (rectangular) coordinates it is

$$\int_0^3 \int_0^{\sqrt{9-x^2}} (2x + 2y)(9 - x^2 - y^2) + x)dydx \quad .$$

One way is to use Maple,

```
int(int((2*x+2*y)*(9-x**2-y**2)+x,y=0..sqrt(9-x**2)),x=0..3);
```

and immediately get $\frac{693}{5}$. If you do it by hand, it is better to convert to polar coordinates. This

$$\int_0^3 \int_0^{\pi/2} (2r(\cos\theta + \sin\theta)(9 - r^2) + r\cos\theta) r \, d\theta$$

and now it is simple (but tedious) calc2 integrals.

Finally since it says **downward** direction, we have to multiply it by -1, getting the answer.

Comments: **1.** Quite a few people got the integrand correctly, but possibly 'adapting' the problem in handout 16.5, took the region to be $0 < x, y < 3$ or other ones. Be careful!

2. Many people did not notice the **downward** word, and did not multiply by -1. This time I was lenient, and only took off a few points, but in the Final, you would get zero points. Read the question! I gave you half an hour to check!

10. (10 points) Find the **point** on the plane $x + 2y + 3z = 18$ where the function $f(x, y, z) = xyz$ is **as large as possible**.

The **type** of the answer is: POINT

---

ans. $(6, 3, 2)$

Sol. We use **Lagrange multipliers**. Here the **goal function** is $f(x, y, z) = xyz$ and the **constraint** is $x + 2y + 3z = 18$ so $g(x, y, z) = x + 2y + 3z$.

We need

$$grad(f) = \lambda \, grad(g)$$

So

$$\langle yz, xz, xy \rangle = \lambda \langle 1, 2, 3 \rangle \quad .$$

Spelling this out, and adding the **constraint** we have the system

$$\{yz = \lambda \quad , \quad xz = 2\lambda \quad , \quad xy = 3\lambda \quad , \quad x + 2y + 3z = 18\}$$

Dividing the equation $yz = \lambda$ by the equation $xz = 2\lambda$, we get $\frac{y}{x} = \frac{1}{2}$.

Dividing the equation $yz = \lambda$ by the eqution $xy = 3\lambda$ we get $\frac{z}{x} = \frac{1}{3}$.

So $y = \frac{1}{2}x$, $z = \frac{1}{3}x$. Using $x + 2y + 3z = 18$ we get

$$x + 2(\frac{1}{2}x) + 3(\frac{1}{3}x) = 18 \quad ,$$

So

$$3x = 18 \quad ,$$

so $x = 6$. Going back to $y = \frac{1}{2}x$, and $z = \frac{1}{3}x$ we get $y = 3$ and $z = 2$.

The solution to the system is $x = 6, y = 3, z = 2$ (and we really don't care about λ that happens to be 6).

So the **desired answer** is that the **point** is $(6, 3, 2)$.

Comment: I was **very disappointed** that many people, including some of the best, gave the anwser 36. This is the **value**, but I asked for the **location** (point).

READ THE QUESTION!

This time I was lenient and gave lots of partial credit for people who did it correctly except that they put 36 as the final answer. Of course they found the values of x, y and z as intermediate steps, but they did not gave them (or rather the point) as the final answer.

You were supposed to use the last half hour to check, not only that you made conceptual or computational errors, but that you answered the **right question** not a different one.

In the Final exam you would get zero points!