SECOND CHANCE CLUB: E1

12.1 Lecture 1

Problems from TB:

43) vector of length 4 in the direction of $u = \langle -1, -1 \rangle$

$|u| = \sqrt{1x^2 + 1x^2} = \sqrt{2x^2} = \pm x\sqrt{2} = 4 \qquad x = \pm \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

$\langle -2\sqrt{2}, -2\sqrt{2} \rangle$

45) vector of length 2 in the direction opposite to $v = i - j$ $\rightarrow \langle 1, -1 \rangle$

$|v| = \sqrt{1x^2 + 1x^2} = \sqrt{2x^2} = \pm x\sqrt{2} = 2 \qquad x = \pm \frac{2}{\sqrt{2}} = \sqrt{2}$

$\langle \sqrt{2}, -\sqrt{2} \rangle$

12.2 Lecture 1

Problems from TB:

33) Find vector parametrization of the line that passes through $P(1,2,-8)$, with a direction vector $v = \langle 2,1,3 \rangle$

$r(t) = \langle 1+2t, 2+t, -8+3t \rangle$

45) which of the following is a parametrization of the line through $P = (4,9,8)$ perpendicular to the xz-plane

(a) $r(t) = \langle 4,9,8 \rangle + t\langle 1,0,1 \rangle$ (b) $r(t) = \langle 4,9,8 \rangle + t\langle 0,0,1 \rangle$

(c) $r(t) = \langle 4,9,8 \rangle + t\langle 0,1,0 \rangle$ (d) $r(t) = \langle 4,9,8 \rangle + t\langle 1,1,0 \rangle$

12.3 Lecture 2

Problems from TB:

59) Find the projection of u along v for $u = \langle a,b,c \rangle$, $v = i$ $\rightarrow \langle 1,0,0 \rangle$

$\langle a,b,c \rangle \cdot \langle 1,0,0 \rangle = ai \qquad \frac{u \cdot v}{|v|} = \frac{ai}{1}$

$\sqrt{1^2} = 1$

61) Compute the component of u along v for $u = \langle 3,2,1 \rangle$, $v = \langle 1,0,1 \rangle$

$\langle 3,2,1 \rangle \cdot \langle 1,0,1 \rangle = 3(1) + 2(0) + 1(1) = 4 \qquad \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \frac{u \cdot v}{|v|} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

12.4 Lecture 2

Problems from TB:

29) Show that if v and w lie in the yz-plane, then $v \times w$ is a multiple of i

$v \to \langle 0, a, b \rangle$

$w \to \langle 0, c, d \rangle$

$$v \times w \quad \begin{vmatrix} i & j & k \\ 0 & a & b \\ 0 & c & d \end{vmatrix} = ((ad)-(bc))i - ((0)(d)-(b)(0))j + ((0)(c)-(a)(0))k$$

$$= (ad-bc)i - 0j + 0k = (ad-bc)i$$

45) Use cross products to find the area of the triangle in the xy-plane defined by $(1,2)$, $(3,4)$, $(-2,2)$

P Q R

$PQ = \langle 3-1, 4-2 \rangle = \langle 2, 2 \rangle$

$PR = \langle -2-1, 2-2 \rangle = \langle -3, 0 \rangle$

$$PQ \times PR \quad \begin{vmatrix} 2 & 2 \\ -3 & 0 \end{vmatrix} = 2(0) - (2)(-3)$$

$$= 0 + 6 = 6 \to \frac{6}{2} = 3$$

$$A = \frac{|PQ \times PR|}{2}$$

12.5 Lecture 3

Problem from a previous Final:

Find an equation for the plane through the point $(1,0,2)$ that contains the line $r(t) = \langle 1,1,1 \rangle + t\langle 1,-1,0 \rangle$. Simplify as much as you can!

$r(t) = \langle 1+t, 1-t, 1 \rangle$

$r(0) = \langle 1, 1, 1 \rangle$

$$1(x-1) + 1(y-0) + 1(z-2) = 0$$

$$x - 1 + y + z - 2 = 0$$

$$x + y + z = 3$$

Similar made-up problem:

Find an equation for the plane through the point $(4,-1,0)$ that contains the line $r(t) = \langle 0,1,2 \rangle + t\langle 1,3,0 \rangle$. Simplify as much as you can!

$r(t) = \langle t, 1+3t, 2 \rangle$

$r(0) = \langle 0, 1, 2 \rangle$

$$0(x-4) + 1(y+1) + 2(z-0) = 0$$

$$y + 1 + 2z = 0$$

$$y + 2z = -1$$

Problems from TB:

39) Find the intersection of the line and plane $x+y+z=14$, $r(t) = \langle 1,1,0 \rangle + t\langle 0,2,4 \rangle$

$r(t) = \langle 1, 1+2t, 4t \rangle$ $\quad 1+1+2t+4t=14 \quad 2+6t=14 \quad 6t=12 \quad t=2 \quad r(2) = (1, 5, 8)$

41) Find the intersection of the line and plane $z=12$, $r(t) = t\langle -6, 9, 36 \rangle$

$r(t) = \langle -6t, 9t, 36t \rangle \quad 36t = 12 \quad t = \frac{1}{3} \quad r(\frac{1}{3}) = (-2, 3, 12)$

13.1 Lecture 4

Problems from TB:

19) $r(t) = \langle \sin t, 0, 4+\cos t \rangle$ traces a circle. Determine the radius, center, and plane

$r(t) = \langle 0, 0, 4 \rangle + 1\langle \sin t, 0, \cos t \rangle$

radius = 1
center = (0,0,4)
plane = xz-plane

31) Use sine and cosine to parametrize the intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$

$\sin^2\theta + \cos^2\theta = 1$

$4x^2 = 0$
$x = 0$
$4\cos^2\theta$

$r(t) = \langle \cos t, \sin t, 4\cos t^2 \rangle$

13.2 Lecture 4

Problem from a previous Final:

Find the velocity and position vectors of a particle whose acceleration is $a(t) = i + j$, and time $t = 0$, the velocity is $i - j$ and position is k

$v(t) = \int a(t) = \int \langle 1, 1, 0 \rangle = \langle t, t, 0 \rangle + C$

$v(0) = \langle 1, -1, 0 \rangle = \langle 0, 0, 0 \rangle + C$

$C = \langle 1, -1, 0 \rangle$

$v(t) = (t+1)i + (t-1)j$

$r(t) = \int v(t) = \int \langle t+1, t-1, 0 \rangle = \langle \frac{t^2}{2} + t, \frac{t^2}{2} - t, 0 \rangle + C$

$r(0) = \langle 0, 0, 1 \rangle = \langle 0, 0, 0 \rangle + C$

$C = \langle 0, 0, 1 \rangle$

$r(t) = (\frac{t^2}{2} + t)i + (\frac{t^2}{2} - t)j + k$

Similar made-up problem:

Find the velocity and position vectors of a particle whose acceleration is $a(t) = 2ti + k$, and time $t = 0$, the velocity is $j - k$ and position is $i + 2j$

$v(t) = \int a(t) = \int \langle 2t, 0, 1 \rangle = \langle t^2, 0, t \rangle + C$

$v(0) = \langle 1, 0, -1 \rangle = \langle 0, 0, 0 \rangle + C$

$C = \langle 1, 0, -1 \rangle$

$v(t) = (t^2+1)i + (t-1)k$

$r(t) = \int v(t) = \int \langle t^2+1, 0, t-1 \rangle = \langle \frac{t^3}{3} + t, 0, \frac{t^2}{2} - t \rangle + C$

$r(0) = \langle 1, 2, 0 \rangle = \langle 1, 0, -1 \rangle + C$

$C = \langle 0, 2, 1 \rangle$

$r(t) = (t^3/3 + t)i + 2j + (t^2/2 - t + 1)k$

Problems from TB:

35) $\frac{d}{dt}(r \times r')$ where $r(t) = \langle t, t^2, e^t \rangle$

$r'(t) = \langle 1, 2t, e^t \rangle$

$$r \times r' = \begin{vmatrix} i & j & k \\ t & t^2 & e^t \\ 1 & 2t & e^t \end{vmatrix} = (t^2(e^t) - 2t(e^t))i - (t(e^t) - 1(e^t))j + (t(2t) - t^2(1))k$$

$$= \langle e^t(t^2-2t), e^t(t-1), t^2 \rangle$$

$$\frac{d}{dt}\langle e^t(t^2-2t), e^t(t-1), t^2 \rangle$$

$$= \langle e^t(t^2-2t) - e^t(2t-2), e^t(t-1) + e^t(1), 2t \rangle$$

$$= \langle t^2e^t - 4te^t + 2e^t, -te^t, 2t \rangle = \langle e^t(t^2-2), -te^t, 2t \rangle$$

39) $\int_{-1}^{3} \langle 8t^2 - t, 6t^3 + t \rangle dt$

$$\frac{8}{3}t^3 - \frac{t^2}{2}\Big|_{-1}^{3} = \frac{216}{3} - \frac{9}{2} - \left(-\frac{8}{3} - \frac{1}{2}\right) = \frac{432 - 27 + 16 + 3}{6} = \frac{424}{6} = \frac{212}{3}$$

$$\frac{6}{4}t^4 + \frac{t^2}{2}\Big|_{-1}^{3} = \frac{486}{4} + \frac{9}{2} - \left(\frac{6}{4} + \frac{1}{2}\right) = \frac{486 + 18 - 6 - 2}{4} = \frac{496}{4} = 124$$

$$\left\langle \frac{212}{3}, 124 \right\rangle$$

13.3 Lecture 5

Problem from a previous Final:

Suppose that the position of a certain particle is given by

$r(t) = \langle e^t\cos t, e^t\sin t, e^t \rangle$, $0 \le t \le \pi$

a) Find the velocity of the particle as a function of the time t

$$v(t) = r'(t) = \langle e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, e^t \rangle$$

$$= \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$$

b) Find the length of the arc traversed by the moving particle for $0 \le t \le \pi$

$\int_0^{\pi} |r'(t)|\, dt$ $\quad r'(t) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$

$$|r'(t)| = \sqrt{(e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2 + (e^t)^2}$$

$$= \sqrt{e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t) + e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t) + e^{2t}}$$

$$= \sqrt{e^{2t}(1 + 1 + 1)} = \sqrt{3e^{2t}} = \sqrt{3}e^t$$

$$\int_0^{\pi} \sqrt{3}e^t\, dt = \sqrt{3}e^t\Big|_0^{\pi} = \sqrt{3}e^{\pi} - \sqrt{3} = \sqrt{3}(e^{\pi} - 1)$$

Similar made-up problem:

Suppose that the position of a certain particle is given by

$r(t) = \langle 2t^2, 4t, 3 \rangle$, $0 \le t \le 2$

a) Find the velocity of the particle as a function of the time t

$v(t) = r'(t) = \langle 4t, 4, 0 \rangle$

b) Find the length of the arc traversed by the moving particle for $0 \le t \le 2$

$\int_0^2 |r'(t)|$ $\quad r'(t) = \langle 4t, 4, 0 \rangle$ $\quad |r'(t)| = \sqrt{16t^2 + 16 + 0} = \sqrt{16(t^2+1)} = 4\sqrt{t^2+1}$

$\int_0^2 4\sqrt{t^2+1}\, dt = 4\left(\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln(t+\sqrt{t^2+1})\right)\Big|_0^2 = 4(\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5}))$

$= 4\sqrt{5} + 2\ln(2+\sqrt{5})$

Problems from TB:

14) Find the speed at the given value of t: $r(t) = \langle e^{t-3}, 12, 3t^{-1} \rangle$, $t=3$

$v(t) = r'(t) = \langle e^{t-3}, 0, -3t^{-2} \rangle \rightarrow v(3) = \langle e^0, 0, -1/3 \rangle = \langle 1, 0, -1/3 \rangle$

16) Find the speed at the given value of t: $r(t) = \langle \cosh t, \sinh t, t \rangle$, $t=0$

$v(t) = r'(t) = \langle -h\sinh t, h\cosh t, 1 \rangle \rightarrow v(0) = \langle 0, h, 1 \rangle$

13.4 Lecture 6

Problem from a Previous Final:

Find the curvature of the curve $r(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ at the point $(1, 1, \frac{2}{3})$

$K(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ $\qquad t=1 \quad t^2=1 \quad \frac{2}{3}t^3 = \frac{2}{3} \rightarrow t=1$

$r'(t) = \langle 1, 2t, 2t^2 \rangle$

$r''(t) = \langle 0, 2, 4t \rangle$

$|r'(t)| = \sqrt{1+4t^2+4t^4}$

$r'(t) \times r''(t)$

$$\begin{vmatrix} i & j & k \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = (8t^2-4t^2)i - (4t-0)j + (2-0)k$$

$= \langle 4t^2, -4t, 2 \rangle$

$|r'(t) \times r''(t)| = \sqrt{16t^4+16t^2+4} = 2\sqrt{4t^4+4t^2+1}$

$K(t) = \dfrac{2\sqrt{4t^4+4t^2+1}}{(\sqrt{1+4t^2+4t^4})^3} \rightarrow K(1) = \dfrac{6}{(3)^3} = \dfrac{6}{27} = \dfrac{2}{9}$

Similar made-up problem:

Find the curvature of the curve $r(t)=\langle 2t^4, 3t, t^2\rangle$ at $t=0$

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$r'(t)=\langle 8t^3, 3, 2t\rangle$

$r''(t)=\langle 24t^2, 0, 2\rangle$

$r'(t) \times r''(t)$

$$\begin{vmatrix} i & j & k \\ 8t^3 & 3 & 2t \\ 24t^2 & 0 & 2 \end{vmatrix} = (6-0)i - (16t^3-48t^3)j + (0-72t^2)k$$

$= \langle 6, 32t^3, -72t^2\rangle$

$|r'(t) \times r''(t)| = \sqrt{36 + 1024t^6 + 5184t^4}$

$|r'(t)| = \sqrt{64t^6+9+4t^2}$

$$K(t) = \frac{\sqrt{36 + 1024t^6 + 5184t^4}}{(\sqrt{64t^6+9+4t^2})^3} \qquad K(0) = \frac{6}{9^3} = \frac{6}{729} = \frac{2}{243}$$

Problems from TB:

17) Find the curvature of the plane curve at the point indicated ; $y=t^4$, $t=2$

$$K(t) = \frac{|y''(t)|}{(1+y'(t)^2)^{3/2}} \Rightarrow K(2) = \frac{48}{(1+(32)^2)^{3/2}} = \frac{48}{(1+32768)} = \frac{48}{32769} \approx 0.0015$$

$y'(t) = 4t^3 \Rightarrow y'(2) = 32$

$y''(t) = 12t^2 \Rightarrow y''(2) = 48$

19) Find the curvature of $r(t)=\langle 2\sin t, \cos 3t, t\rangle$ at $t=\frac{\pi}{3}$ and $t=\frac{\pi}{2}$

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$r'(t) = \langle 2\cos t, -3\sin 3t, 1\rangle$

$r''(t) = \langle -2\sin t, -9\cos 3t, 0\rangle$

$|r'(t)| = \sqrt{4\cos^2 t + 9\sin^2 3t + 1}$

$|r'(t) \times r''(t)|$

$$\begin{vmatrix} i & j & k \\ 2\cos t & -3\sin 3t & 1 \\ -2\sin t & -9\cos 3t & 0 \end{vmatrix} = (0+9\cos 3t)i - (0+6\sin t\sin 3t)j + (0+9\cos 3t)k$$

$= \langle 9\cos 3t, -6\sin t\sin 3t, 9\cos 3t\rangle$

$\sqrt{81\cos^2 3t + 36\sin^2 t\sin^2 3t + 81\cos^2 3t}$

$$K(t) = \frac{\sqrt{81\cos^2 3t + 36\sin^2 t\sin^2 3t + 81\cos^2 3t}}{(\sqrt{4\cos^2 T + 9\sin^2 3t + 1})^3} \Rightarrow$$

$K(\frac{\pi}{3}) = 4.54$

$K(\frac{\pi}{2}) = 0.2$

13.5 Lecture 5

Problems from TB:

33) Find the coefficients a_T and a_N as a function of t ; $r(t)=\langle t, \cos t, \sin t\rangle$

$v(t) = r'(t) = \langle 1, -\sin t, \cos t\rangle$

$a(t) = r''(t) = \langle 0, -\cos t, -\sin t\rangle$

$$a_T = \frac{\langle 1, -\sin t, \cos t\rangle \cdot \langle 0, -\cos t, -\sin t\rangle}{\sqrt{1^2 + \sin^2 t + \cos^2 t}}$$

$$a_T = \frac{0 + \sin t\cos t - \sin t\cos t}{\sqrt{2}} = 0$$

$v(t) \times a(t)$

$$\begin{vmatrix} i & j & k \\ 1 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = (\sin^2 t + \cos^2 t)i - (-\sin t - 0)j + (-\cos t + 0)k$$

$\langle 1, \sin t, -\cos t\rangle$

$\sqrt{1+\sin^2 t + \cos^2 t} = \sqrt{1+1} = \sqrt{2}$

$$a_R = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

35) Find the coefficients a_T and a_N as a function of t ; $r(t) = \langle e^{2t}, t, e^{-t} \rangle$, $t=0$

$v(t) = r'(t) = \langle 2e^{2t}, 1, -e^{-t} \rangle$

$a(t) = r''(t) = \langle 4e^{2t}, 0, e^{-t} \rangle$

$$a_T = \frac{\langle 2e^{2t}, 1, -e^{-t} \rangle \cdot \langle 4e^{2t}, 0, e^{-t} \rangle}{\sqrt{4e^{4t} + 1^2 + e^{-2t}}}$$

$$a_T = \frac{8e^{4t} + 0 - e^{-t}}{\sqrt{4 + 1 + 1}} \Rightarrow a_T(0) = \frac{7}{\sqrt{6}}$$

$v(t) \times a(t)$

$$\begin{vmatrix} i & j & k \\ 2e^{2t} & 1 & -e^{-t} \\ 4e^{2t} & 0 & e^{-t} \end{vmatrix} = (e^{-t} - 0)i - (2e^{2t}e^{-t} + 4e^{2t}e^{-t})j + (0 - 4e^{2t})k = \langle e^{-t}, -6e^{2t}e^{-t}, -4e^{2t} \rangle$$

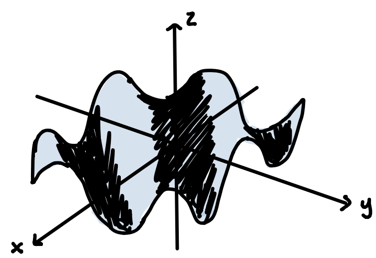
$$\sqrt{e^{-2t} + 36e^{4t}e^{-2t} + 16e^{4t}} = \sqrt{53}$$

$$a_N = \frac{\sqrt{53}}{\sqrt{6}}$$

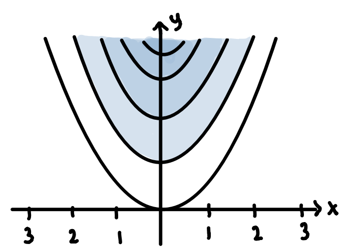
14.1 Lecture 6

Problems from TB:

25) Sketch the graph and draw several vertical and horizontal traces: $f(x,y) = \sin(x-y)$



29) Draw a contour map of $f(x,y) = x^2 - y$



14.2 Lecture 6

Problem from a Previous Finally:

Compute the limit $\lim_{(x,y,z)\to(1,1,1)} e^{-xy}\sin(\pi z/2)$ or prove it does not exist

$$\lim_{(x,y,z)\to(1,1,1)} e^{-xy}\sin(\pi z/2) = e^{-1}\sin(\pi/2) = e^{-1}(1) = \frac{1}{e}$$

Similar made-up problem:

Compute the limit $\lim_{(x,y,z)\to(1,1,1)} xz^2+4y+3$ or prove it does not exist

$$\lim_{(x,y,z)\to(1,1,1)} xz^2+4y+3 = 3$$

Problems from TB:

29) Evaluate the limit or determine that it does not exist: $\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}}$

$$\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}} = \frac{0}{\sqrt{4}} = 0$$

33) Evaluate the limit or determine that it does not exist: $\lim_{(x,y)\to(1,-3)} e^{x-y}\ln(x-y)$

$$\lim_{(x,y)\to(1,-3)} e^{x-y}\ln(x-y) = e^{1+3}\ln(1+3) = e^4\ln(4)$$

14.3 Lecture 7

Problems from TB:

37) Compute the first-order partial derivatives: $w = xy^2z^3$

$$\frac{\partial w}{\partial x} = y^2z^3 \quad \frac{\partial w}{\partial y} = 2xyz^3 \quad \frac{\partial w}{\partial z} = 3xy^2z^2$$

41) Compute the given partial derivatives: $f(x,y) = 3x^2y + 4x^3y^2 - 7xy^5$, $f_x(1,2)$

$f_x = 6xy + 12x^2y^2 - 7y^5 \quad f_x(1,2) = 6(1)(2) + 12(1)^2(2)^2 - 7(2)^5 = 12+48-224 = -164$

14.4 Lecture 7

Problem from a Previous Final:

Find an equation of the tangent plane to the surface $z = e^{2x-3y}$ at the point $(3,2,1)$. Simplify as much as you can.

$f_x = 2e^{2x-3y} \rightarrow f_x(3,2) = 2e^0 = 2$

$f_y = -3e^{2x-3y} \rightarrow f_y(3,2) = -3e^0 = -3$

$z-1 = 2(x-3) - 3(y-2)$

$z = 2x-6-3y+6-1$

$z = 2x-3y-1$

Similar made-up problem:

Find an equation of the tangent plane to the surface $z = xy^2 - y^3$ at the point $(1,4,2)$. Simplify as much as you can.

$f_x = y^2 \longrightarrow f_x(1,4) = 16$

$f_y = 2xy - 3y^2 \rightarrow f_y(1,4) = 8-48 = -40$

$z-2 = 16(x-1) - 40(y-4)$

$z = 16x - 16 - 40y + 160 + 2$

$z = 16x - 40y + 146$

$z = 8x - 20y + 73$

Problems from TB:

29) Suppose that the plane tangent to $z=f(x,y)$ at $(-2,3,4)$ has equation $4x+2y+z=2$. Estimate $f(-2.1, 3.1)$

$\hookrightarrow z=-4x-2y+2$

$f_x=-4$ $\Rightarrow (-2,3) \Rightarrow$ $f_x=-4$

$f_y=-2$ $\qquad\qquad\qquad$ $f_y=-2$

$z-4=-4(x+2)-2(y-3)$

$z=-4(-2.1+2)-2(3.1-3)+4$

$z=-4(-0.1)-2(0.1)+4 = 0.4-0.2+4 = 4.2$

$f(-2.1, 3.1) \approx 4.2$

31) A boy has weight $W=34$ kg and height $H=1.3$ m. Use the linear approximation to estimate the change in I if (W,H) changes to $(36, 1.32)$

$I=\frac{W}{H^2}$ $\quad \frac{\partial I}{\partial W}=\frac{1}{H^2}$ $\quad \frac{\partial I}{\partial H}=\frac{-2W}{H^3}$

$\frac{\partial I}{\partial H}\Big|_{(36,1.32)} = \frac{1}{1.32^2} = 0.5917$

$\frac{\partial I}{\partial W}\Big|_{(36,1.32)} = \frac{-2(36)}{1.32^3} = -30.9512$

$\Delta W = 36-34=2$ $\qquad \Delta H = 1.32-1.3=0.02$

$\Delta I = 0.5917(2)-30.9512(0.02)$

$= 0.564376$

14.5 Lecture 8

Problem from a Previous Final:

Let $f(x,y,z) = -x^2+y^2+z^2-1$

a) Compute ∇f $\rightarrow$ $\nabla f = \langle -2x, 2y, 2z\rangle$

b) Find a normal to the level surface $f(x,y,z)=0$ at the point $(1,1,1)$ and give an equation for the tangent plane to that surface at that point

normal: $\nabla f(1,1,1) = \langle -2,2,2\rangle$

$2(z-1)=-2(x-1)+2(y-1)$

$2z-2=-2x+2+2y-2$

$2z=-2x+2y+2$

$z=-x+y+1$

c) Compute the directional derivative of $f(x,y,z)$ at the point $(1,1,1)$ in the direction $\langle 1,2,2\rangle$

$\nabla f = \langle -2x, 2y, 2z\rangle \rightarrow \nabla f(1,1,1) = \langle -2,2,2\rangle$

$|u| = \sqrt{1+4+4} = \sqrt{9} = 3$

$u = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$

$\nabla f(1,1,1)\cdot u = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle \cdot \langle -2,2,2\rangle$

$= -\frac{2}{3}+\frac{4}{3}+\frac{4}{3} = \frac{6}{3} = 2$

similar made-up problem:

Let $f(x,y,z) = 2x + 3y^3 + zy$

a) Compute ∇f → $\nabla f = \langle 2, 9y^2, y \rangle$

b) Find a normal to the level surface $f(x,y,z) = 0$ at the point $(1,1,1)$ and give an equation for the tangent plane to that surface at that point

normal: $\nabla f(1,1,1) = \langle 2, 9, 1 \rangle$ $\quad 1(z-1) = 2(x-1) + 9(y-1)$ $\quad z = 2x + 9 - 10$

$z - 1 = 2x - 2 + 9y - 9$

c) Compute the directional derivative of $f(x,y,z)$ at the point $(1,1,1)$ in the direction $\langle 1,3,2 \rangle$

$\nabla f = \langle 2, 9y^2, y \rangle \rightarrow \nabla f(1,1,1) = \langle 2, 9, 1 \rangle$

$|u| = \sqrt{1+9+4} = \sqrt{14}$

$u = \langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle$

$\nabla f(1,1,1) \cdot u = \langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle \cdot \langle 2, 9, 1 \rangle$

$= \frac{2}{\sqrt{14}} + \frac{27}{\sqrt{14}} + \frac{2}{\sqrt{14}} = \frac{31}{\sqrt{14}} = \frac{31\sqrt{14}}{14}$

Problems from TB:

29) Calculate the directional derivative in the direction of v at the given point

$g(x,y,z) = xe^{-yz}$, $v = \langle 1,1,1 \rangle$, $P = (1,2,0)$

$\nabla g = \langle e^{-yz}, -xe^{-yz}, -xe^{-yz} \rangle \rightarrow \nabla g(1,2,0) = \langle e^0, -e^0, -e^0 \rangle = \langle 1, -1, -1 \rangle$

$|v| = \sqrt{1+1+1} = \sqrt{3}$

$u = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$\nabla g \cdot u = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$

31) Find the directional derivative of $f(x,y) = x^2 + 4y^2$ at the point $P = (3,2)$ in the direction pointing to the origin

$\nabla f = \langle 2x, 8y \rangle \rightarrow \nabla f(3,2) = \langle 6, 16 \rangle$

$\nabla f \cdot u = \langle 6, 16 \rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$

$|v| = \sqrt{9+4} = \sqrt{13}$ $\quad u = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$

$= \frac{18}{\sqrt{13}} + \frac{32}{\sqrt{13}} = \frac{50}{\sqrt{13}} \rightarrow -\frac{50}{\sqrt{13}}$

14.6 Lecture 9

Problem from a Previous Final:

Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$ as functions of r and s, if $f(x,y) = x^3 + 2xy + y^3$ and the variables are related by $x = r - s$ and $y = r + s$. You do not need to simplify!

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}\right)$

$= (3x^2 + 2y)(1) + (2x + 3y^2)(1)$

$= 3x^2 + 2x + 3y^2 + 2y$

$= 3(r-s)^2 + 2(r-s) + 3(r+s)^2 + 2(r+s)$

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}\right)$

$= (3x^2 + 2y)(-1) + (2x + 3y^2)(1)$

$= -3x^2 + 2x + 3y^2 - 2y$

$= -3(r-s)^2 + 2(r-s) + 3(r+s)^2 - 2(r+s)$

Another Problem from a Previous Final:

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+2y+3z)=5xyz+1$

$\hookrightarrow \sin(x+2y+3z)-5xyz-1=0$

$(\cos(x+2y+3z))(1+0+3\frac{\partial z}{\partial x})-(5yz+\frac{\partial z}{\partial x}5xy)=0$

$\cos(x+2y+3z)+3\frac{\partial z}{\partial x}(\cos(x+2y+3z))-5yz-\frac{\partial z}{\partial x}5xy=0$

$\frac{\partial z}{\partial x}(3\cos(x+2y+3z)-5xy)=-\cos(x+2y+3z)+5yz$

$$\frac{\partial z}{\partial x}=\frac{5yz-\cos(x+2y+3z)}{3\cos(x+2y+3z)-5xy}$$

$(\cos(x+2y+3z))(0+2+3\frac{\partial z}{\partial y})-(5xz+\frac{\partial z}{\partial y}5xy)=0$

$2\cos(x+2y+3z)+3\frac{\partial z}{\partial y}\cos(x+2y+3z)-5xz-\frac{\partial z}{\partial y}5xy=0$

$\frac{\partial z}{\partial y}(3\cos(x+2y+3z)-5xy)=5xz-2\cos(x+2y+3z)$

$$\frac{\partial z}{\partial y}=\frac{5xz-\cos(x+2y+3z)}{3\cos(x+2y+3z)-5xy}$$

Similar made-up Problem #1:

Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$ as functions of r and s, if $f(x,y)=\cos(xy)+x^2$ and the variables are related by $x=r+s$ and $y=r^2+2s$

$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}\right)+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}\right)$

$=(-\sin(xy)+2x)(1)+(-\sin(xy))(2r)$

$=-\sin((r+s)(r^2+2s))-2r\sin((r+s)(r^2+2s))$

$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}\right)+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}\right)$

$=(-\sin(xy)+2x)(1)+(-\sin(xy))(2)$

$=-\sin((r+s)(r^2+2s))-2\sin((r+s)(r^2+2s))$

Similar made-up Problem #2:

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\cos(x+y+z)-xyz=5$

$\hookrightarrow \cos(x+y+z)-xyz-5=0$

$(-\sin(x+y+z))(1+0+\frac{\partial z}{\partial x})-(yz+\frac{\partial z}{\partial x}xy)=0$

$-\sin(x+y+z)-\frac{\partial z}{\partial x}\sin(x+y+z)-yz-\frac{\partial z}{\partial x}xy=0$

$\frac{\partial z}{\partial x}(-\sin(x+y+z)-xy)=yz+\sin(x+y+z)$

$$\frac{\partial z}{\partial x}=\frac{yz+\sin(x+y+z)}{-\sin(x+y+z)-xy}$$

$(-\sin(x+y+z))(0+1+\frac{\partial z}{\partial y})-(xz+\frac{\partial z}{\partial y}xy)=0$

$-\sin(x+y+z)-\frac{\partial z}{\partial y}\sin(x+y+z)-xz-\frac{\partial z}{\partial y}xy=0$

$\frac{\partial z}{\partial y}(-\sin(x+y+z)-xy)=xz+\sin(x+y+z)$

$$\frac{\partial z}{\partial y}=\frac{xz+\sin(x+y+z)}{-\sin(x+y+z)-xy}$$

Problems from TB:

25) suppose that z is defined implicitly as a function of x and y by the equation
$F(x,y,z) = xz^2 + y^2z + xy - 1 = 0$

a) calculate F_x, F_y, F_z

$F_x = z^2 + y$ $\quad$ $F_y = 2yz + x$ $\quad$ $F_z = 2xz + y^2$

b) calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z^2+y}{2xz+y^2}$ $\qquad$ $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2yz+x}{2xz+y^2}$

26) calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the points $(3,2,1)$ and $(3,2,-1)$, where z is defined implicitly by the equation $z^4 + z^2x^2 - y - 8 = 0$

$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{2xz^2}{4z^3+2x^2z}$ $\qquad$ $\left.\frac{\partial z}{\partial x}\right|_{(3,2,1)} = \frac{-6}{22} = -\frac{3}{11}$ $\qquad$ $\left.\frac{\partial z}{\partial x}\right|_{(3,2,-1)} = \frac{6}{22} = \frac{3}{11}$

$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = \frac{1}{4z^3+2x^2z}$ $\qquad$ $\left.\frac{\partial z}{\partial y}\right|_{(3,2,1)} = \frac{1}{22}$ $\qquad$ $\left.\frac{\partial z}{\partial y}\right|_{(3,2,-1)} = \frac{1}{-22}$

14.7 Lecture 10

Problem from a Previous Final:

Find the local maximum and minimum points, the local maximum and minimum values, and saddle points of the function $f(x,y) = 4x^2 + y^2 + 2x^2y - 1$

$f_x = 8x + 4xy \rightarrow f_{xx} = 8 + 4y$
$\qquad\qquad\qquad\quad f_{xy} = 4x$
$f_y = 2y + 2x^2 \longrightarrow f_{yy} = 2$

$f_x = 8x + 4xy = 0$
$x(8+4y) = 0$
$x = 0 \quad y = -2$

$f_y = 2y + 2x^2 = 0$
$2(y + x^2) = 0$
$y = -x^2$

$(0,0) \quad (-\sqrt{2},-2) \quad (\sqrt{2},-2)$

saddle Points:
$(\sqrt{2},-2)$
$(-\sqrt{2},-2)$
Local min at
$(0,0)$

$f_{xx}(\pm\sqrt{2},-2) = 2$
$f_{xy}(\pm\sqrt{2},-2) = 4\sqrt{2}$ $\quad D = (2)(2) - (4\sqrt{2})^2 = 4 - 32 = -28$
$f_{yy}(\pm\sqrt{2},-2) = 2$

$f_{xx}(0,0) = 8$
$f_{xy}(0,0) = 0$ $\quad D = (8)(2) - 0^2$
$f_{yy}(0,0) = 2$ $\quad\quad = 16$

Another Problem from a Previous Final:

Find the local maximum and minimum point(s), the local maximum and minimum values, and saddle point(s) of the function $f(x,y) = 6y^2 - 2y^3 + 3x^2 + 6xy$

$f_x = 6x + 6y \longrightarrow f_{xx} = 6$
$\qquad\qquad\qquad\quad f_{xy} = 6$
$f_y = 12y - 6y^2 + 6x \rightarrow f_{yy} = 12 - 12y$

$f_{yy}(0,0) = 12 \quad f_{yy}(-1,1) = 0$

$f_x = 6x + 6y = 0$
$y = -x$

$D = 6(12) - 36 = 36$
$D = 6(0) - 36 = -36$

$f_y = 12y - 6y^2 + 6x = 0$
$y^2 - 2y = x$
$y(y-2) = x$
$y = x \quad y = x + 2$
$(-1,1) \quad (0,0)$

$-12x - 6x^2 + 6x = 0$
$-6x - 6x^2 = 0$
$-6x(x+1) = 0$
$x = 0 \quad x = -1$

saddle point $(-1,1)$
local min $(0,0)$

Similar made-up Problem #1:

Find the local maximum and minimum points, the local maximum and minimum values, and saddle points of the function $f(x,y) = 2x^2 + y^2 + x^2y - 1$

$f_x = 4x + 2xy \rightarrow f_{xx} = 4 + 2y$

$f_y = 2y + x^2$ $\quad f_{xy} = 2x$ $\quad f_{yy} = 2$

$f_x = 4x + 2xy = 0$

$2x(2+y) = 0$

$x = 0 \quad y = -2$

$f_y = 2y + x^2 = 0$

$2y + x^2 = 0$

$x = 0,\ y = 0 \mid y = -2,\ x = 2$

$f_{xx}(0,-2) = 0$, $f_{xy}(0,-2) = 0$, $D = (0)(2) - (0) = +2$

$f_{xx}(0,0) = 4$, $f_{xy}(0,0) = 0$, $= 8(2) - 0 = +\#$

$f_{xx}(2,-2) = 0$, $f_{xy}(2,-2) = 8$, $= 0(2) - (64) = -\#$

local min: $(0,0)$

saddle point: $(2,-2)$

Similar made-up Problem #2:

Find the local maximum and minimum point(s), the local maximum and minimum values, and saddle point(s) of the function $f(x,y) = x^2 + y^2 + xy$

$f_x = 2x + y \rightarrow f_{xx} = 2$

$f_{xy} = 1$

$f_y = 2y + x \rightarrow f_{yy} = 2$

$f_x = 2x + y = 0$ $\quad f_y = 2y + x = 0$

$$\begin{array}{r} -4x - \cancel{2y} = 0 \\ x + \cancel{2y} = 0 \\ \hline -3x = 0 \end{array}$$

$x = 0$, $y = 0$

$f_{xx}(0,0) = 2$, $f_{xy}(0,0) = 0$, $f_{yy}(0,0) = 0$

$D = 2(0) - (0)^2 = 2$

minimum at $(0,0)$

Problems from TB:

31) Determine the global extreme values of the function on the given set without using calculus: $f(x,y) = (x^2+y^2+1)^{-1}$, $0 \le x \le 3$, $0 \le y \le 5$

$0 \le x^2 \le 9$

$0 \le y^2 \le 25$

$0 \le x^2 + y^2 \le 34 \rightarrow 1 \le x^2 + y^2 + 1 \le 35$

$\frac{1}{35}$

33) Show that $f(x,y) = xy$ does not have a global minimum or a global maximum on the domain $D = \{(x,y): 0 < x < 1, 0 < y < 1\}$

$f_x = y$ $\quad f_y = x$

$y = 0$ $\quad x = 0$

$f_{xx} = y$, $f_{xy} = 1$, $f_{yy} = x$ $\rightarrow (0,0) = 0,\ 1,\ 0$

$D = (0)(0) - 1^2 = -\#$

only a saddle point at $(0,0)$

The End