

11/22/20

"Problems from a Previous Final" L1-10

12.5 $r(t) = \langle 1, 1, 1 \rangle + t \langle 1, -1, 0 \rangle ; (1, 0, 2)$

$PQ = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$

$PR = \langle 1, 0, -1 \rangle$

$PQ \times PR = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = i \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$

$= i + j + k \Rightarrow \langle 1, 1, 1 \rangle$

$1(x-1) + 1(y-1) + 1(z-1) = 0 \Rightarrow x+y+z=3$

13.2 $a(t) = i + j ; v(0) = i - j, s(0) = k$

$v(t) = \int a(t) dt = \int i + j dt = ti + tj + C \quad (C = i - j)$

$r(t) = \int v(t) dt = \int ti + tj + i - j dt = (\frac{1}{2}t^2 + t)i + (\frac{1}{2}t^2 - t)j + C$

$v(t) = (t+1)i + (t-1)j ; r(t) = (t^2/2 + t)i + (t^2/2 - t)j + k$

13.3 $r(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle, 0 \leq t \leq \pi$

(a) $v(t) = \text{deriv.}(r(t)) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$

(b) $\text{arc length} = \int_0^\pi |r'(t)| dt = \sqrt{3}(e^\pi - 1)$

13.4 $r(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ at pt. $(1, 1, \frac{2}{3}) \Rightarrow t=1$

$r'(t) = \langle 1, 2t, 2t^2 \rangle \Rightarrow r'(1) = \langle 1, 2, 2 \rangle$

$r''(t) = \langle 0, 2, 4t \rangle \Rightarrow r''(1) = \langle 0, 2, 4 \rangle$

$r'(1) \times r''(1) = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} = i \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$
 $= 4i - 4j + 2k$

$|r'(1) \times r''(1)| = \sqrt{4^2 + (-4)^2 + 2^2} = \sqrt{16 + 16 + 4} = \sqrt{36} = 6$

$|r'(1)| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

$k(t) = \frac{|r'(1) \times r''(1)|}{|r'(1)|^3} = \frac{6}{27} = \frac{2}{9}$

14.2 $\lim_{(x,y,z) \rightarrow (1,1,1)} e^{-xy} \sin(\pi z/2) = e^{-(1)(1)} \sin(\pi(1)/2) = e^{-1} \sin(\pi/2) = e^{-1} = 1/e$

14.4 $z = e^{2x-3y}$ at pt. $(3, 2, 1)$

$f_x = 2e^{2x-3y} \Rightarrow f_x(3, 2) = 2$

$f_y = -3e^{2x-3y} \Rightarrow f_y(3, 2) = -3$

$(z-1) = 2(x-3) - 3(y-2)$

$z = 2x - 6 - 3y + 6 + 1 \Rightarrow z = 2x - 3y + 1$

14.5 $f(x,y,z) = -x^2 + y^2 + z^2 - 1$

(a) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle -2x, 2y, 2z \rangle$

(b) $f(x,y,z) = 0$ at pt. $(1,1,1) \Rightarrow z = x - y + 1$

(c) $\| \langle 1, 2, 2 \rangle \| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

$\nabla f \cdot u = \langle -2, 2, 2 \rangle \cdot \frac{\langle 1, 2, 2 \rangle}{3} = \frac{-2 + 4 + 4}{3} = \frac{6}{3} = 2$

14.6 $f(x,y) = x^3 + 2xy + y^3$; $x = r - s$ & $y = r + s$

$f_r = (f_x)(x_r) + (f_y)(y_r) = (3x^2 + 2y)(1) + (2x + 3y^2)(1) = 3x^2 + 2y + 2x + 3y^2$

$f_s = (f_x)(x_s) + (f_y)(y_s) = (3x^2 + 2y)(-1) + (2x + 3y^2)(1) = -3x^2 - 2y + 2x + 3y^2$

$df/dr = 3(r-s)^2 + 2(r+s) + 3(r+s)^2 + 2(r-s)$

$df/ds = -3(r-s)^2 - 2(r+s) + 3(r+s)^2 + 2(r-s)$

14.6 $\sin(x + 2y + 3z) = 5xyz + 1 \Rightarrow F(x,y,z) = \sin(x + 2y + 3z) - 5xyz - 1$

$\frac{dz}{dx} = -\frac{F_x}{F_z} = \frac{(5yz - \cos(x + 2y + 3z))}{(3\cos(x + 2y + 3z) - 5xy)}$

$\frac{dz}{dy} = -\frac{F_y}{F_z} = \frac{(5xz - 2\cos(x + 2y + 3z))}{(3\cos(x + 2y + 3z) - 5xy)}$

14.7 $f(x,y) = 4x^2 + y^2 + 2x^2y - 1$

$f_x(x,y) = 8x + 4xy = 0$ (1)

$f_y(x,y) = 2y + 2x^2 = 0$ (2) $y = -x^2$ ^{sub.}

$8x + 4x(-x^2) = 0$

$y = -(0)^2 = 0$

$8x - 4x^3 = 0$

$y = -(\sqrt{2})^2 = -2$

$4x(2 - x^2) = 0$

$y = -(-\sqrt{2})^2 = -2$

$x = 0, \pm\sqrt{2}$

crit. pts. : $(0,0), (\sqrt{2}, -2), (-\sqrt{2}, -2)$

$f_{xx}(x,y) = 8 + 4y$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (8 + 4y)(2) - (4x)^2$

$f_{yy}(x,y) = 2$

$= 16 + 8y - 16x^2 = -16x^2 + 8y + 16$

$f_{xy}(x,y) = 4x$

$D(0,0) = 16 > 0 \Rightarrow f_{xx}(0,0) = 8 > 0$ (local min.) $\Rightarrow f(0,0) = -1$

$D(\sqrt{2}, -2) = -32 < 0 \Rightarrow$ saddle pt.

$D(-\sqrt{2}, -2) = -32 < 0 \Rightarrow$ saddle pt.

14.7 $f(x,y) = 6y^2 - 2y^3 + 3x^2 + 6xy$

Local max : none. Local Min : location : $(0,0)$, value : 0. Saddle pt : $(-1,1)$

Book Exercises (2 per section) L1-10

11/22/20

12.1: #43, 49

43. Vector of length 4 in the direction of $u = \langle -1, -1 \rangle$

$$eu = \frac{1}{\|u\|} u = \frac{1}{\sqrt{(-1)^2 + (-1)^2}} \langle -1, -1 \rangle = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$4eu = 4 \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{4}{\sqrt{2}}, -\frac{4}{\sqrt{2}} \right\rangle = \langle -2\sqrt{2}, -2\sqrt{2} \rangle$$

49. Find all scalars λ s.t. $\lambda(2, 3)$ has length 1.

$$\|\lambda \langle 2, 3 \rangle\| = |\lambda| \|\langle 2, 3 \rangle\| = |\lambda| \sqrt{2^2 + 3^2} = |\lambda| \sqrt{13}$$

$$|\lambda| \sqrt{13} = 1 \Rightarrow \lambda_1 = \frac{1}{\sqrt{13}}, \lambda_2 = -\frac{1}{\sqrt{13}}$$

$$|\lambda| = 1/\sqrt{13}$$

12.2: #37, 39

37. Passes through $(1, 1, 1)$ & $(3, -5, 2)$

$$r(t) = (1-t)\vec{OP} + t\vec{OQ} \quad ; \quad \vec{OP} = \langle 1, 1, 1 \rangle \text{ \& \ } \vec{OQ} = \langle 3, -5, 2 \rangle$$

$$r(t) = (1-t)\langle 1, 1, 1 \rangle + t\langle 3, -5, 2 \rangle = \langle 1+2t, 1-6t, 1+t \rangle$$

39. Passes through O & $(4, 1, 1)$

$$r(t) = (1-t)\langle 0, 0, 0 \rangle + t\langle 4, 1, 1 \rangle = \langle 0, 0, 0 \rangle + \langle 4t, t, t \rangle = \langle 4t, t, t \rangle$$

12.3: #61, 75

61. $u = \langle 3, 2, 1 \rangle$, $v = \langle 1, 0, 1 \rangle$

$$u \cdot v = 4 \quad ; \quad v \cdot v = \|v\|^2 = 1^2 + 1^2 = 2 \Rightarrow \left\| \frac{(u \cdot v)}{(v \cdot v)} v \right\| = \frac{4}{2} \|v\| = 2\sqrt{2}$$

75. Calc. projection of \vec{AC} along \vec{AD} : \vec{DC} is \perp to the face OAD of the cube. Hence, it is orthogonal to the segment \vec{AD} on this face. Therefore, the projection of the vector \vec{AC} along \vec{AD} is the vector \vec{AD} itself.

12.4: #57, 73

57. Show that 3 pts. P, Q, R are collinear if & only if $\vec{PQ} \times \vec{PR} = 0$.

They lie on 1 line if \vec{PQ} & \vec{PR} are parallel. By basic props. of cross product $\Rightarrow \vec{PQ} \times \vec{PR} = 0$

73. Prove that $v \times w = v \times u$ if & only if $u = w + \lambda v$ for some scalar λ . ($v \neq 0$)

$$v \times w = v \times u$$

$$0 = v \times u - v \times w$$

$$0 = v \times (u - w)$$

$$u - w = \lambda v$$

$$u = w + \lambda v$$

12.5: # 43, 45

43. $3x - 9y + 4z = 5$, yz

The yz -plane has the eq. $x=0$, hence the intersection of the plane w/ the yz -plane must satisfy both $x=0$ & the eq. of the plane $3x - 9y + 4z = 5$. That is, this is the set of all pts. $(0, y, z)$ in the yz -plane s.t. $-9y + 4z = 5$.

45. $3x + 4z = -2$, xy

The trace of the plane $3x + 4z = -2$ in the xy coordinate plane is the set of all pts. that satisfy the eq. of the plane & the eq. $z=0$ of the xy coordinate plane. Thus, we sub. $z=0$ in $3x + 4z = -2$ to obtain the line $3x = -2$ or $x = -2/3$ in the xy -plane.

13.1: # 15, 19

15. (A) \Rightarrow (ii) ; (C) \Rightarrow (i) ; (B) \Rightarrow (iii)

19. $r(t) = \langle \sin t, 0, 4 + \cos t \rangle \Rightarrow x(t) = \sin t, z(t) = 4 + \cos t$
 $x^2 + (z-4)^2 = \sin^2 t + \cos^2 t = 1$ (traces \bigcirc of radius 1, centered at $(0,0,4)$)

13.2: # 51, 53

51. $r''(t) = 16k$, $r(0) = \langle 1, 0, 0 \rangle$, $r'(0) = \langle 0, 1, 0 \rangle$

$$r'(t) = \int r''(t) dt = \int 16k dt = (16t)k + c_1$$

$$r(t) = \int r'(t) dt = \int ((16t)k + c_1) dt = \left(\int 16(t) dt \right) k + c_1 t + c_2 = (8t^2)k + c_1 t + c_2$$

$$r'(0) = c_1 = \langle 0, 1, 0 \rangle = j$$

$$r(0) = 0k + c_1 \cdot 0 + c_2 = \langle 1, 0, 0 \rangle \Rightarrow c_2 = \langle 1, 0, 0 \rangle = i$$

$$r(t) = (8t^2)k + tj + i = i + tj + (8t^2)k$$

53. $r''(t) = \langle 0, 2, 0 \rangle$, $r(3) = \langle 1, 1, 0 \rangle$, $r'(3) = \langle 0, 0, 1 \rangle$

$$r'(t) = \int r''(t) dt = \int \langle 0, 2, 0 \rangle dt = \langle 0, 2t, 0 \rangle + c_1$$

$$r(t) = \int r'(t) dt = \int (\langle 0, 2t, 0 \rangle + c_1) dt = \langle 0, t^2, 0 \rangle + c_1 t + c_2$$

$$r'(3) = \langle 0, 6, 0 \rangle + c_1 = \langle 0, 0, 1 \rangle \Rightarrow c_1 = \langle 0, -6, 1 \rangle$$

$$r(3) = \langle 0, 9, 0 \rangle + c_1(3) + c_2 = \langle 1, 1, 0 \rangle$$

$$\langle 0, 9, 0 \rangle + \langle 0, -18, 3 \rangle + c_2 = \langle 1, 1, 0 \rangle \Rightarrow c_2 = \langle 1, 10, -3 \rangle$$

$$r(t) = \langle 0, t^2, 0 \rangle + t \langle 0, -6, 1 \rangle + \langle 1, 10, -3 \rangle$$

$$= \langle 1, t^2 - 6t + 10, t - 3 \rangle$$

13.3: # 17, 19

17. $y = x^2$; at pt. $(1,1)$, speed = 500 km/h

$$m = \left. \frac{dy}{dx} \right|_{x=1} = \left. \frac{d(x^2)}{dx} \right|_{x=1} = 2x \Big|_{x=1} = 2 \Rightarrow \langle 1, 2 \rangle \text{ is direction vector of tan. line at } x=1$$

$$500 = \|\lambda \langle 1, 2 \rangle\| = |\lambda| \|\langle 1, 2 \rangle\| = \sqrt{5} |\lambda| \Rightarrow \lambda = \pm 100\sqrt{5}$$

Tan. vector: $100\sqrt{5} \langle 1, 2 \rangle$

19. $\int_0^T r'(u) du = r(T) - r(0) \Rightarrow r(T) = r(0)$

$\int_0^T \|r'(u)\| du = \text{length of the path traveled by the bee in the time interval } 0 \leq t \leq T.$

13.4: # 29, 31

29. $\langle t^2, t^3 \rangle$, $t=2 \Rightarrow x(t) = t^2$, $y(t) = t^3$

$$x'(t) = 2t, \quad y'(t) = 3t^2$$

$$x''(t) = 2, \quad y''(t) = 6t$$

$$\Rightarrow \begin{aligned} x'(2) &= 4, & y'(2) &= 12 \\ x''(2) &= 2, & y''(2) &= 12 \end{aligned}$$

$$k(2) = \frac{|x'(2)y''(2) - x''(2)y'(2)|}{(x'(2)^2 + y'(2)^2)^{3/2}} = \frac{|4 \cdot 12 - 2 \cdot 12|}{(4^2 + 12^2)^{3/2}} = \frac{24}{160^{3/2}} \approx 0.012$$

31. $\langle t \cos t, \sin t \rangle$, $t = \pi \Rightarrow x(t) = t \cos t$, $y(t) = \sin t$

$$x'(\pi) = -1, \quad y'(\pi) = -1$$

$$x''(\pi) = \pi, \quad y''(\pi) = 0$$

$$k(\pi) = \frac{|x'(\pi)y''(\pi) - x''(\pi)y'(\pi)|}{(x'(\pi)^2 + y'(\pi)^2)^{3/2}} = \frac{|-1 \cdot 0 - \pi \cdot (-1)|}{((-1)^2 + (-1)^2)^{3/2}} = \frac{\pi}{2\sqrt{2}} \approx 1.11$$

13.5: # 27, 29

27. $F = \langle 5, 2 \rangle$ acts on a 10-kg mass; find $r(10)$ if $v_0 = \langle 2, -3 \rangle$

$$F = ma \Rightarrow a = \langle 0.5, 0.2 \rangle \Rightarrow v = \langle 0.5t + 2, 0.2t - 3 \rangle$$

$$\Rightarrow r = \langle 0.25t^2 + 2t, 0.1t^2 - 3t \rangle$$

$$\Rightarrow r(10) = \langle 45, -20 \rangle$$

29. (a) if avg. velocity = 0, then particle must be back at $t = T$

$$\Rightarrow \bar{v} = \frac{1}{T} \int_0^T r'(t) dt = r(t) \Big|_0^T$$

(b) avg. speed need not $\neq 0$! Ex: $r(t) = \langle \cos t, \sin t \rangle$. from 0 to 2π , avg. velocity = 0, but constant avg. speed = 1

14.1: #45, 47

45. $\Delta d (\theta \text{ to } C) = 0.0005 \text{ Kg/m}^3$; $\Delta S (\theta \text{ to } C) = 33.6 - 32.7 = 0.9 \text{ ppt.}$

Avg. ROC from B to C = $\Delta d / \Delta S = 0.0005 / 0.9 = 0.000556 \text{ Kg/m}^3 \cdot \text{ppt}$

47. The 2 adjacent level curves are closer to the level curve of A than the corresponding 2 adjacent level curves are to the level curve of B. This suggests that water density is more sensitive to a change in temp. at A than B.

14.2: #33, 37

33. $\lim_{(x,y) \rightarrow (\pi, 0)} \frac{\cos x}{\sin y} = \text{DNE}$

37. $\lim_{(x,y) \rightarrow (-3,-2)} (x^2 y^3 + 4xy) = 9(-8) + 4(-3)(-2) = -72 + 24 = -48$

14.3: #41, 43

41. $f(x,y) = 3x^2y + 4x^3y^2 - 7xy^5$, $f_x(1,2) = 6 \cdot 1 \cdot 2 + 12 \cdot 1^2 \cdot 2^2 - 7 \cdot 2^5 = -164$
 $f_x(x,y) = 6xy + 12x^2y^2 - 7y^5$

43. $g(u,v) = u \ln(u+v)$, $g_u(1,2) = \ln(1+2) + 1/(1+2) = \ln 3 + 1/3$
 $g_u(u,v) = d/du (u \ln(u+v)) = 1 \cdot \ln(u+v) + u \cdot 1/(u+v) = \ln(u+v) + u/(u+v)$

14.4: #29, 31

29. $z = f(x,y)$ at $(-2, 3, 4)$ has eq. $4x + 2y + z = 2 \Rightarrow z = 2 - 4x - 2y$
 $f(-2.1, 3.1) \approx 2 - 4(-2.1) - 2(3.1) = 4.2$

31. $W = 34 \text{ kg}$ & $H = 1.3 \text{ m}$

$$I(34+h, 1.3+k) - I(34, 1.3) \approx dI/dW(34, 1.3)h + dI/dH(34, 1.3)k$$

$$\Delta I \approx dI/dW(34, 1.3) \cdot 2 + dI/dH(34, 1.3) \cdot 0.02$$

$$dI/dW = d/dW W/H^2 = 1/H^2 \Rightarrow dI/dW(34, 1.3) = 0.5917$$

$$dI/dH = W d/dH H^{-2} = W \cdot (-2H^{-3}) = -2W/H^3 \Rightarrow dI/dH(34, 1.3) = -30.9513$$

$$\Delta I \approx 0.5917 \cdot 2 - 30.9513 \cdot 0.02 = 0.5644$$

14.5: #53, 57

53. Find a fn. $f(x,y,z)$ s.t. $\nabla f = \langle z, 2y, x \rangle$

$$f(x,y,z) = xz + y^2$$

57. Est. $\Delta f = f(3.53, 8.98) - f(3.5, 9)$ assuming that $\nabla f(3.5, 9) = \langle 2, -1 \rangle$
 $\Delta f \approx \nabla f_p \cdot \Delta \mathbf{v} \Rightarrow \Delta \mathbf{v} = \langle 3.53 - 3.5, 8.98 - 9 \rangle = \langle 0.03, -0.02 \rangle$
 $\approx \nabla f(3.5, 9) \cdot \Delta \mathbf{v} = \langle 2, -1 \rangle \cdot \langle 0.03, -0.02 \rangle = 0.08$

14.6: 35, 43

35. Compute $\nabla(1/r) \Rightarrow$ let $F(r) = 1/r \Rightarrow F'(r) = -1/r^2$
 $\nabla(1/r) = F'(r) \mathbf{e}_r = -1/r^2 \cdot \mathbf{r}/\|\mathbf{r}\| = -1/r^3 \cdot \mathbf{r}$

43. Show that $f(x)$ is differentiable & $c \neq 0$ is a constant, then $u(x, t) = f(x - ct)$ satisfies the so-called advection eq.

$$du/dt + c du/dx = 0 \quad s = x - ct, \quad u(x, t) = f(s)$$

$$du/dt = f'(s) ds/dt = f'(s) \cdot (-c) = -c f'(s)$$

$$du/dx = f'(s) ds/dx = f'(s) \cdot 1 = f'(s)$$

$$du/dt = -c du/dx \quad \text{or} \quad du/dt + c du/dx = 0$$

14.7: #33, 43

33. The largest & smallest vals. of f on the closed square $0 \leq x, y \leq 1$ are $f(1, 1) = 2$ & $f(-1, -1) = -2$. However, on the open square $0 < x, y < 1$, f can never attain these max. & min. values, since the boundary (& in particular pts. $(1, 1)$ & $(-1, -1)$) is not included in the domain. This doesn't contradict Thm. 3 since the domain is open.

43. $f(x, y) = x^2 + 2y^2, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$

$f(x, y)$ is max. at $(1, 1)$

is min. if $x = y = 0 \quad (0, 0)$

$$\text{Global max} = f(1, 1) = 1^2 + 2 \cdot 1^2 = 3$$

$$\text{Global min} = f(0, 0) = 0^2 + 2 \cdot 0^2 = 0$$