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MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)

Do not write below this line

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1. (out of 12)
2. (out of 12)
3. (out of 12)
4. (out of 12)
5. (out of 12)
6. (out of 12)
7. (out of 12)
8. (out of 12)
9. (out of 12)
10. (out of 12)
11. (out of 12)
12. (out of 12)
13. (out of 12)
14. (out of 12)
15. (out of 12)
16. (out of 12)
17. (out of 8)

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tot. (out of 200)

Important note: Unlike Exams 1 and 2, you are not required to state the type of the answer, and there is no credit for stating the type. But if the given answer is the **wrong type**, you would get 0 points.

Example: Find $f'(2)$ if $f(x) = x^3$. If you give the answer $3x^2$ instead of 12, you would get **zero** points!

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane, D, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 points) Compute the line-integral

$$\oint_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: 200 Pi

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1. $$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(7y)$$

$$= -4$$

$$\iint_D (-4)\, dA$$

$$D = \{(x,y) \mid y = -\sqrt{100-x^2} \,..\, \sqrt{100-x^2}\ ,\ x = 0..10\}$$

$$-\int_0^{10} \int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}} (-4)\, dy\, dx$$

$$= -\int_0^{10} \left(-8 \cdot \sqrt{100-x^2}\right) dx$$

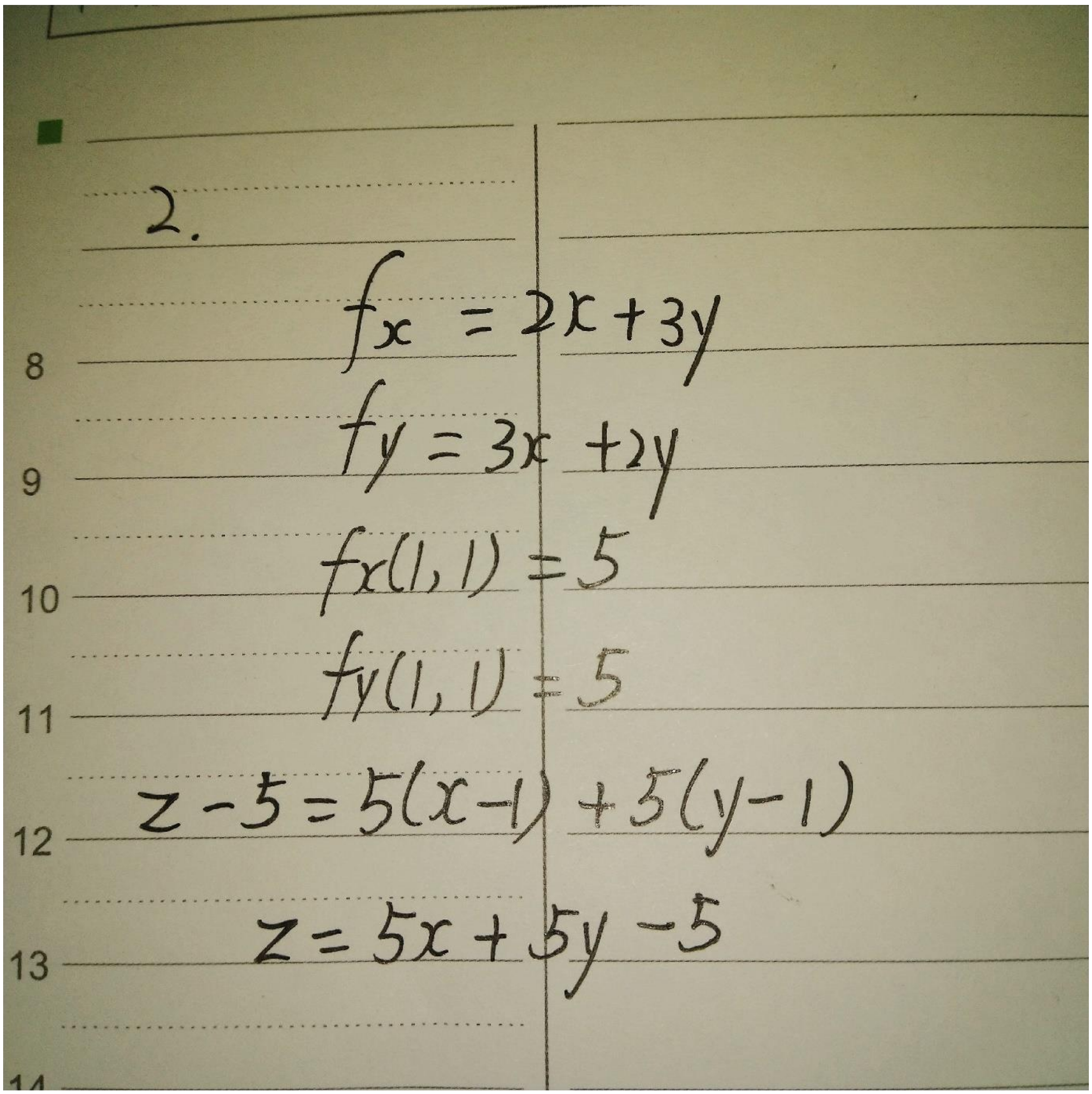
$$= 200\pi$$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point (1, 1,5).

Ans.: z=**5x+5y-5**



2.

$$f_x = 2x + 3y$$

$$f_y = 3x + 2y$$

$$f_x(1,1) = 5$$

$$f_y(1,1) = 5$$

$$z - 5 = 5(x-1) + 5(y-1)$$

$$z = 5x + 5y - 5$$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2 y$ in the region

$$\{(x, y) \mid 0 \le x \le 1 \,,\, 0 \le y \le 1 - x\}.$$

Absolute minimum value:

0

Absolute maximum value:

2/3

3.

$f_x = 2xy \qquad f_y = x^2$

$\begin{cases} 2xy = 0 \\ x^2 = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \end{cases}$

$f(0,0) = 0$

$\{x=0,\ 0 \le y \le 1-x\}$

$f = 0$

$\{x=1,\ 0 \le y \le 1-x\}$

$f = y = 0$

$\{y=0,\ 0 \le x \le 1\}$

$f = 0$

$\{y=1-x,\ 0 \le x \le 1\}$

$f = x^2(1-x)$

$f' = 2x - 3x^2 = 0$

$x = 0$ or $x = \frac{2}{3}$

$f(0) = 0,\ f(\frac{2}{3}) = \frac{2}{3}$

min. 0, max. $\frac{2}{3}$

4. (12 points) Compute $f_{xxyz}(0, 0, 0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x, y, z)\big|_{x=0, y=0, z=0}$) if

$$f(x, y, z) = \sin(x^2 + y + z) \quad .$$

Ans.: -2

4.

$f_{xxyz}(0,0,0)\,(\sin(x^2+y+z))$

$= f_{xyz}(0,0,0)\,(2x \cdot \cos(x^2+y+z))$

$= f_{yz}(0,0,0)\,(2\cos(x^2+y+z) - 4x^2 \cdot \sin(x^2+y+z))$

$= f_{z}(0,0,0)\,(-4 \cdot \cos(y+x^2+z) \cdot x^2 - 2\sin(y+x^2+z)$

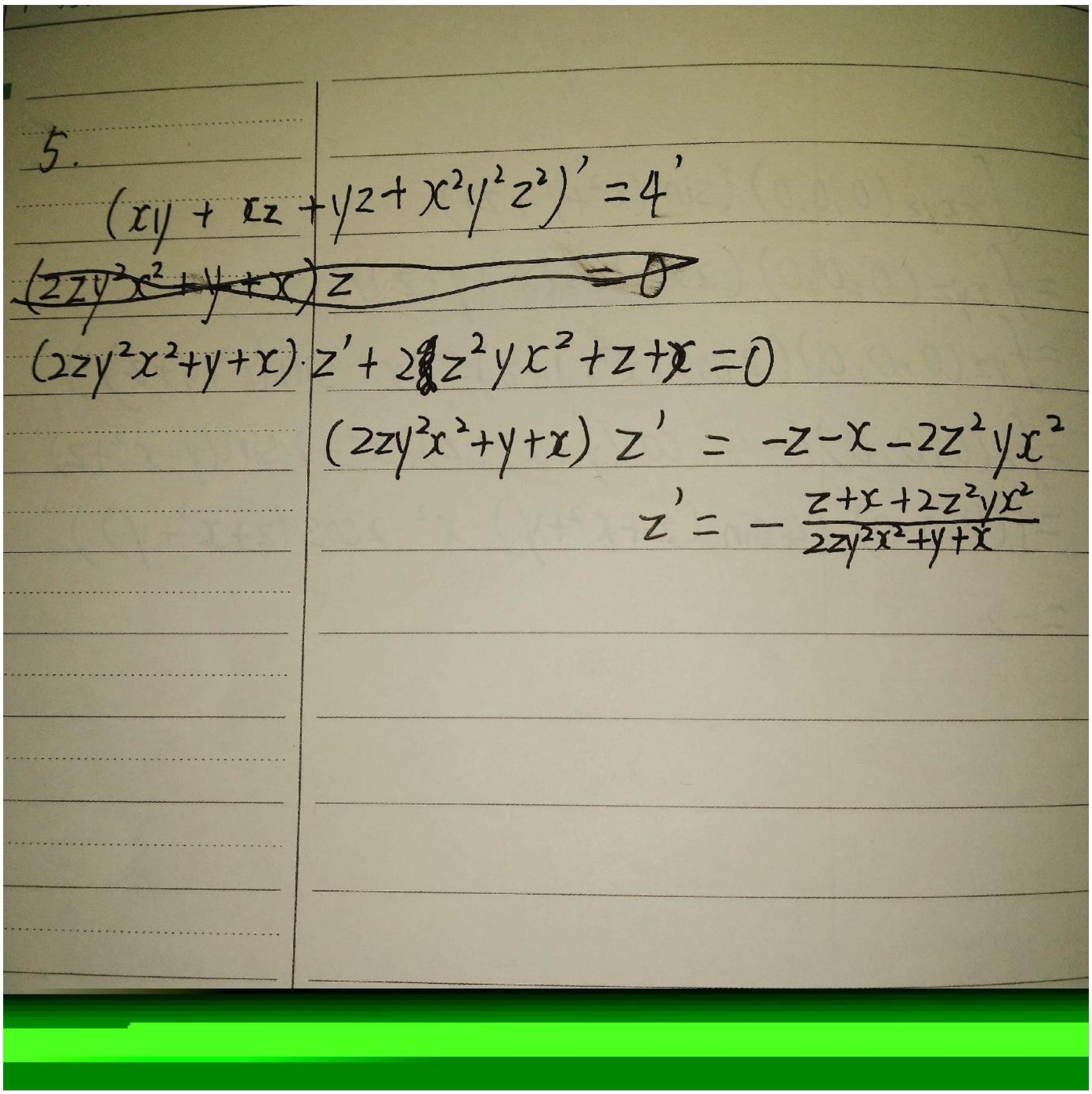
$= f(0,0,0)\,(4\sin(z+x^2+y) \cdot x^2 - 2\cos(z+x^2+y))$

$= -2$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point (1，1，1) if (x, y, z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: －（z+x+2z^2*y*x^）/(2z*y^2*x^2+y+x)



5.

$$(xy + xz + yz + x^2y^2z^2)' = 4'$$

$$\cancel{(2zy^2x^2+y+x)z = 0}$$

$$(2zy^2x^2+y+x)\cdot z' + 2z^2yx^2 + z + x = 0$$

$$(2zy^2x^2+y+x)\, z' = -z-x-2z^2yx^2$$

$$z' = -\frac{z+x+2z^2yx^2}{2zy^2x^2+y+x}$$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t,\ y = 2 + t,\ z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

and the line

$$x = -t,\ y = 1 + t,\ z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: N/A

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6. ~~$x_t = 1$, $y_t = 1$, $z_t = 1$~~

line 1 : $x_t = 1$, $y_t = 1$, $z_t = 1$

line 2 : $x_t = -1$, $y_t = 1$, $z_t = 1$

$\langle 1, 1, 1\rangle \times \langle -1, 1, 1\rangle = \langle 0, -2, 2\rangle$

$$\begin{cases} 1+t = -t \\ 2+t = 1+t \\ 3+t = 2+t \end{cases}$$

Since there are no solutions, and line 1 is not parellel to line 2, the two lines are on different surface.

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4\sin 2t, -4\cos 2t, 9e^{3t} \rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3\rangle$ and its position at $t = 0$ is $\langle 0, 1, 1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: (1,0,e^(3/4Pi))

7.

$$V(t) = \int a(t)\,dt = \langle 2\cos(2t)+C_1, -2\sin(2t)+C_2, 3e^{3t}+C_3$$

$$\cancel{2\cos(2t)}\ 2\cos(0) + C = 2$$

$$C = 0$$

$$V(t) = \langle 2\cos(2t), -2\sin(2t), 3e^{3t}\cancel{+C} \rangle$$

$$\int_0^{\frac{\pi}{4}} V(t)\,dt = \langle 1, -1, e^{\frac{3}{4}\pi} - 1 \rangle$$

$$\text{Position} = \langle 0, 1, 1 \rangle + \langle 1, -1, e^{\frac{3}{4}\pi} - 1 \rangle$$

$$= \langle 1, 0, e^{\frac{3}{4}\pi} \rangle$$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2\, z)\; ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = (t, 2t, 2t) \quad , \quad 0 \leq t \leq 1 \quad .$$

Ans.: 13/2

8.

$$\int_C (x+y+2z)\, ds$$

$$= \int_C (t + 2t + 4t)\, dS$$

$$\cancel{= \int_0^1} \cdot \quad r'(t) = \langle 1, 2, 2 \rangle$$

$$\int_0^1 t\, dt + \int_0^1 2t \cdot 2dt + \int_0^1 4t \cdot 2dt$$

$$= 13/2$$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3} f(x,y,z)\right) \cos\left(\frac{\pi}{4} g(x,y,z)\right)$$

Ans.: 0

9.

$$\lim_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3} f(x,y,z)\right) \cos\left(\frac{\pi}{4} g(x,y,z)\right)$$

$$= \lim_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{2}\right)$$

$$\doteq 0$$

10. (12 points) Compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = (x^2 + \sin(y + z)\ ,\ y^2 + xz^3\ ,\ z^2 + e^{xy}\ )$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x, y, z) \,|\, 0 \leq x, y, z \leq 1\}$$

with the normal pointing **outward**.

Ans.: 3

10.

$$\operatorname{div} F = \frac{\partial}{\partial x}\left(x^2+\sin(y+z)\right) + \frac{\partial}{\partial y}\left(y^2+xz^3\right) + \frac{\partial}{\partial z}\left(z^2+e^{xy}\right)$$

$$= 2x + 2y + 2z$$

$$\{(x,y,z) \,|\, 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$$

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

$$= \int_0^1\int_0^1\int_0^1 2x+2y+2z \ dz\,dy\,dx$$

$$= \int_0^1\int_0^1 2x+2y+1 \ dy\,dx$$

$$= \int_0^1 2x+2 \ dx$$

$$= 3$$

11. (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F}\, d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) = \left(2e^{2x+3y+4z},\ 3e^{2x+3y+4z},\ 4e^{2x+3y+4z} \right) \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: e^**12 -1**

11.

$f_x = 2e^{2x+3y+4z}$

$f = e^{2x+3y+4z} + g(y,z)$

$f_y = 3e^{2x+3y+4z}$

$f = e^{2x+3y+4z} + g(z)$

$f_z = 4e^{2x+3y+4z}$

$f = e^{2x+3y+4z}$

~~plug in t~~ $r(t) = \langle t, 2t, t^2 \rangle$

$\cancel{f = e^{8t+4t^2}}$ $r(0) = \langle 0, 0, 0 \rangle$

$\cancel{\int_C F \cdot dr}$ $r(1) = \langle 1, 2, 1 \rangle$

$\int_C F \cdot dr = f(1,2,1) - f(0,0,0)$

$= e^{12} - 1$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2$, $y = t$, $z = t^2$, $0 \le t \le 1$.

Ans.: 8

12.

$dx = 2t\,dt$

$dy = 1\,dt$

$dz = 2t\,dt$

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz$$

$$= \int_0^1 5t \cdot 2t\,dt + \int_0^1 5t^2\,dt + \int_0^1 6t^2 \cdot 2t\,dt$$

$$= 8$$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV \quad ,$$

where E is the hemisphere

$$\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 100, z < 0\} \quad .$$

Ans.: -200Pi

13.

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV \quad \{(x,y,z) \mid x^2+y^2+z^2 \leq 100, z<0$$

$$= \int_{0}^{2\pi}\int_{\pi}^{2\pi}\int_0^{10} \frac{1}{\rho}\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi}^{2\pi} 50 \cdot \sin(\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi} -100 \; d\theta$$

$$= -200\pi$$

14. (12 points) Evaluate the quadruple integral

$$\iiiint_E 360\,x\,dV \quad ,$$

where

$$E = \{(x, y, z, w)\,|\,0 \le w \le 1\,,\ 0 \le z \le w\,,\ 0 \le y \le z\,,\ 0 \le x \le y\} \quad .$$

Ans.: 3

14.

$$\iiiint_E 360x\,dV$$

$$= \int_0^1 \int_0^w \int_0^z \int_0^y 360x\,dx\,dy\,dz\,dw$$

$$= \int_0^1 \int_0^w \int_0^z 180y^2\,dy\,dz\,dw$$

$$= \int_0^1 \int_0^w 60z^3\,dz\,dw$$

$$= \int_0^1 15w^4$$

15. (12 points) Find the Jacobian of the transformation from $(u,\ v)$-space to $(x,\ y)$-space.

$$x = 3\sin(2u+v) \quad , \quad y = u + v + \cos(u+v) \quad ,$$

at the point $(u,\ v) = (0,\ 0)$.

Ans.: -3

15.

$$\frac{\partial x}{\partial u} = \cancel{6\sin(2u+v)}\ 6\cos(2u+v)$$

$$\frac{\partial x}{\partial v} = \cancel{3\sin(v+}\ 3\cos(v+2u)$$

$$\frac{\partial y}{\partial u} = 1-\sin(u+v)$$

$$\frac{\partial y}{\partial v} = 1-\sin(v+u)$$

$$\begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 6\cos(2u+v), & 3\cos(v+2u) \\ 1-\sin(u+v), & 1-\sin(v+u) \end{vmatrix}$$

Jacobian: $3\cos(v+2u)(1-\sin(u+v)) - 6\cos(2u+v)(1-\sin(v+u))$

$$= 3(\sin(u+v)-1)\cdot\cos(2u+v)$$

Plugin (u,v)

$$3(\sin(0)-1)\cdot\cos(0) = -3$$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s):

Local minimum points(s): (6,18)

saddle point(s): (0,0)

16. $f_x = 3x^2 - 6y$

$f_y = 2y - 6x$

$f_{xx} = 6x$

$f_{xy} = -6$

$f_{yy} = 2$

$\begin{cases} 3x^2 - 6y = 0 \\ 2y - 6x = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$ $\begin{cases} x = 6 \\ y = 18 \end{cases}$

criticle point : $(0,0)$ $(6,18)$

$f_{xx}(0,0) = 0$ $f_{xy}(0,0) = -6$ $f_{yy}(0,0) = 2$

$D = 0 - 36 = -36 < 0$

$(0,0)$ is a saddle point.

$f_{xx}(6,18) = 36$ $f_{xy}(6,18) = -6$ $f_{yy}(6,18) = 2$

$D = 72 - 36 = 36 > 0$

$f_{xx}(6,18) > 0$

$(6,18)$ is a local minimum

15.

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\,\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = (x + y \;\; y + z\,,\; x + z) \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z)\,|\,(x-1)^2 + (y+2)^2 + (z-4)^2 = 100\} \quad .$$

Ans. 4000Pi

17.

$$\text{div}\, F = 1 + 1 + 1 = 3$$

$$\iint_S F \cdot dS = 3 \times \frac{4}{3} \times \pi r^3$$

$$= 4000\pi$$