Q26 Rahul Paleja

1. Compute the line integral

$$\int_C 7y\,dx + 3x\,dy$$

where C is the circle $x^2+y^2=100$ traveled in the clockwise direction

$\Rightarrow$ closed curve

Answer: Greene's Theorem $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= \int_C 7y\,dx + 3x\,dy = \iint_R (3-7)\,dA = -4\iint_R dA = \underset{\substack{\uparrow \\ \text{Clockwise}}}{-}(-4 \cdot \text{Area}(R))$

$A = \pi r^2 \quad r = 10$

$= \pi\, 100 = (100\pi)(4) = \boxed{400\pi}$

2. Find An Equation of the tangent plane to the surface

$z = x^2 + 3xy + y^2$

at the point $(1,1,5)$

Answer: $f_x = 2x + 3y \qquad F_x(1,1) = 5$

$f_y = 3x + 2y \qquad F_y(1,1) = 5$

$z - 5 = 5(x-1) + 5(y-1) + 5$
$+5$

$z = 5x - 5 + 5y - 5 + 5 = z = 5x + 5y - 5$

$\boxed{z = 5(x+y-1)}$

3. Find the absolute max value and abs. min value of the function $F(x,y) = x^2y$ in the region

$\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$f_x = 2xy = 0 \quad F_y = x^2 = 0 \qquad (0,0)$ is in the region
$y = 0 \qquad x = 0$

$F(0,0) = 0$

Left Side: $x = 0;\ 0 \le y \le 1-x \quad F(0,y) = 0 = F(y)$

$F'(y) = 0 \qquad F(0) = 0$

End Points: $F(0) = 0 \quad F(1-x) = 0 \qquad$ Abs. Min On LS: 0

Abs. Max On LS: 0

Right Side: $f(1,y) = y = F(y)$ $F'(y) = 1$
$F(1) = 1$ Abs Max: 1
$F(0) = 0$ $F(1-x) = 1-x$ Abs. Min: 0
Down Side: $f(x,0) = 0 = F(x)$ $F'(x) = 0$
$F(0) = 0$ $F(0) = 0$ $F(1) = 0$
Abs. Min = 0   Abs. Max = 0
Upside: $F(x, 1-x) = x(1-x) = (x - x^2) = F(x)$ $F'(x) = 1 - 2x = 0$
$x = 1/2$
$F(1/2) = 1/4$
$F(0) = 0$ $F(1) = 0$
Abs. Max = $1/4$ Abs. Min = 0

Abs. Min Value = 0   Abs. Max Value = 1

(4) Compute $f_{xxyz}(0,0,0)$ if $f(x,y,z) = \sin(x^2+y+z)$

$f_x = \cos(x^2+y+z) \cdot 2x$

$f_{xx} = 4x^2\cos(x^2+y+z) + 2\cos(x^2+y+z)$ $u = 2x$, $u' = 2$ $v = \cos(x^2+y+z)$, $v' = 2x\cos(x^2+y+z)$

$f_{xxy} = -4x^2\sin(x^2+y+z) - 2\sin(x^2+y+z)$

$f_{xxyz} = -4x^2\cos(x^2+y+z) - 2\cos(x^2+y+z)$

$f_{xxyz}(0,0,0) = 0 - 2\cos(0) = 0 - 2(1) = \boxed{-2}$

(5) Point lies on surface $\frac{dz}{dy}$ at point $(1,1,1)$ of $xy + xz + yz + x^2y^2z^2 = 4$

$dz/dy = x + xz' + (yz' + z)$ $u = y$, $u' = 1$ $v = z$, $v' = z'$ $u = y^2$, $u' = 2y$ $v = z^2$, $v' = 2zz'$

$+ x^2(2zy^2z' + 2yz^2) = 0$

$= x + xz' + yz' + z + 2x^2zy^2z' + 2x^2yz^2 = 0$

$z'(x + y + 2x^2zy^2) = -x - z - 2xyz^2$

$z' = \dfrac{-x - z - 2xyz^2}{x + y + 2x^2zy^2}$

$dz/dy(1,1,1) = \dfrac{-1-1-2}{1+1+2} = \dfrac{-4}{4} = \boxed{-1}$

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6. Find an Equation for the Plane that contains both the line $x=1+t, y=2+t, z=3+t$ $(-\infty<t<\infty)$ and the line $x=-t, y=1+t, z=2+t$ $(-\infty<t<\infty)$

Answer: Parametric Forms Of Both Lines:

$\langle 1,2,3\rangle + \langle 1,1,1\rangle t$

$\langle 0,1,2\rangle + \langle -1,1,1\rangle t$

Cross Product for Normal Vector:

$n = \langle 1,1,1\rangle \times \langle -1,1,1\rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \quad i(1-1) - j(1-(-1)) + k(1-(-1))$$

$= \langle 0,-2,2\rangle$

Point: $t=0 \Rightarrow (1,2,3) \Rightarrow$ From 1st Equation

$0(x-1) - 2(y-2) + 2(z-3) = 0$

$-2y+4+2z-6=0$

$-2y+2z-2=0$

$-2y+2z=2 \qquad \dfrac{2(-y+z)}{2} = \dfrac{2}{2} \qquad \boxed{-y+z=1}$

7. $a(t) = \langle -4\sin 2t, -4\cos 2t, 9e^{3t}\rangle$ if its velocity at $t=0$ is $\langle 2,0,3\rangle$ and its position at $t=0$ is $\langle 0,1,1\rangle$, Find its position at time $t=\pi/4$

$\int a(t) = v(t) = \int -4\sin(2t)i - 4\cos(2t)j + 9e^{3t}k\,dt$

$-4\int \sin(2t)\,i$ $\qquad u=3t$, $du=3\,dt$

$u=2t$

$\dfrac{du}{2} = \dfrac{2\,dt}{2}$

$2\cos(2t)i - 2\sin(2t)j + 3e^{3t}k + C = v(t)$

$v(0) = 2i - 0j + 3e^{3}k + C = 2i + 0j + 3k$

$C = 0i + 0j + (3-3e^{3})k$

$v(t) = 2\cos(2t)i - 2\sin(2t)j + (3e^{3t} + 3 - 3e^{3})k$

$P(t) = \sin(2t)i + \cos(2t)j + (3te^{3t} + 3t - 3te^{3})k + C$

$0i + \cancel{j} + 0k + C = 0i + \cancel{j} + k$

$C = k$

$P(t) = \sin(2t)\,i + \cos(2t)\,j + (3te^{3t} + 3t - 3te^3 + 1)K$

$P(\frac{\pi}{4}) = i + 0\,j + (\frac{3\pi}{4} e^{3\pi/4} + \frac{3\pi}{4} - \frac{3\pi}{4} e^3 + 1) \cdot K$

$= \langle 1, 0, \frac{3\pi}{4} e^{3\pi/4} + \frac{3\pi}{4} - \frac{3\pi}{4} e^3 + 1 \rangle$

(8) Compute line integral of:

$\int_C (x+y+2z)\,ds$ where curve C is given by $r(t) = \langle t, 2t, 2t \rangle,\ 0 \le t \le 1$

Answer: $|r'(t)| = \langle 1, 2, 2 \rangle\,dt$

$= \sqrt{1^2+2^2+2^2}\,dt = \sqrt{9} = 3\,dt = ds$

$\int_0^1 (t + 2t + 4t)\,3\,dt = 3\frac{t^2}{2} + 3t^2 + 6t^2 \Big]_0^1$

$= \frac{3}{2} + 3 + 6 = \frac{(2)9}{(2)1} + \frac{3}{2} = \boxed{\frac{21}{2}}$

(9) 0

(10) Compute $\iint_S F \cdot ds$ where $F = \langle x^2 + \sin(y+z), y^2 + xz^3, z^2 + e^{xy} \rangle$

and where S is the boundary (w/ all 6 faces) of the cube $\{(x,y,z) \mid 0 \le x, y, z \le 1\}$ with the normal pointing outward.

Answer: Divergence Theorem $\to$ Surface Integral over a closed surface

$\iint_S F\,ds = \iiint_E (\operatorname{div} F)\,dV$ $\quad \operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= 2x + 2y + 2z$

$\int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\,dz\,dy\,dx$

Inside: $\int_0^1 2x + 2y + 2z\,dz = (2x+2y)z + z^2 \Big]_0^1$

$= (2x+2y) + 1$

Middle: $\int_0^1 (2x+2y) + 1\,dy = 2xy + y^2 + y \Big]_0^1 = 2x + 2$

Outer: $\int_0^1 2x + 2\,dx = x^2 + 2x \Big]_0^1 = 1 + 2 = \boxed{3}$

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(11) We are given that F is conservative as we are given the Df, $F(x,y,z) = e^{2x+3y+4z}$ from inspection. Because F is conservative, we can use the Fundamental Theorem of Line Integrals.

with $t=0$, $x=0$; $y=0$; $z=0$ so Start Pt. is $(0,0,0)$
End Point is with $t=1$; $(1,2,1)$

$$F(\text{end}) - F(\text{start}) = F(1,2,1) - F(0,0,0)$$

$$= e^{2+6+4} - e^0 = \boxed{e^{12} - 1}$$

(12) Evaluate $\int_C 5y\,dx + 5x\,dy + 6z\,dz$

$C:\ x = t^2,\ y = t,\ z = t^2,\ 0 \le t \le 1$
$dx = 2t\,dt \quad dy = dt \quad dz = 2t\,dt$

$$\int_0^1 5t(2t)dt + 5(t^2)dt + 6(t^2)(2t)dt$$

$$= \int_0^1 10t^2 + 5t^2 + 12t^3\,dt$$

$$= \frac{10t^3}{3} + \frac{5t^3}{3} + 3t^4 \Big]_0^1 = \boxed{8}$$

(13) Evaluate $\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$

where E is the hemisphere
$\{(x,y,z) \mid x^2+y^2+z^2 \le 100,\ z \le 0\}$

Spherical Coordinates: $\int_{\pi/2}^{0} \int_{\pi/2}^{0} \int_0^{10} \rho \sin\phi \, d\rho\, d\theta\, d\phi$

(14) $\iiiint_E 360x\,dV \quad E = \{(x,y,z,w) \mid 0 \le w \le 1, 0 \le z \le w, 0 \le y \le z, 0 \le x \le y\}$

$$\int_0^1 \int_0^w \int_0^z \int_0^y 360x\,dx\,dy\,dz\,dw$$

Inner:

$$360\left(\frac{x^2}{2}\right]_0^y = 180y^2$$

1st Middle:

$$\int_0^z 180y^2\,dy = 180\frac{y^3}{3}\Big]_0^z = 60z^3$$

2nd Middle:

$$\int_0^w 60z^3\,dz = 60\left(\frac{z^4}{4}\right]_0^w = 15w^4$$

Outer:

$$\int_0^1 15w^4\,dw = 15\left(\frac{w^5}{5}\right]_0^1 = \boxed{3}$$

(15) Find jacobian of:

$x = 3\sin(2u+v)$, $y = u+v+\cos(u+v)$

at $(u,v) = (0,0)$

$$\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 6\cos(2u+v) & 3\cos(2u+v) \\ 1-\sin(u+v) & 1-\sin(u+v) \end{vmatrix}$$

Plug in $(0,0)$

$$\begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix} = 6-3 = \boxed{3}$$

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(16) $f(x,y) = x^3 + y^2 - 6xy$; Find local max, mins, saddle pts:

Answer:

$f_x = 3x^2 - 6y = 0$ $\quad 3x^2 - 18x = 0 \quad 3x(x-6) = 0$

$x = 0, 6$

$f_y = 2y - 6x = 0 \quad y = 3x$ $\quad$ Critical Pts: $(0,0)$ $(6,18)$

$f_{xx} = 6x$

$f_{xy} = -6$

$f_{yy} = 2$

$f_{xx}(0,0) = 0$

$f_{xy}(0,0) = -6$

$f_{yy}(0,0) = 2$

$D = f_{xx} \cdot f_{yy} - |f_{xy}|^2$

$= 0 - 36 = -36$

$(0,0) \to$ saddle point

$(6,18)$ $\quad f_{xx}(6,18) = 36$

$f_{xy}(6,18) = -6$

$f_{yy}(6,18) = 2$

$36(2) - |-6|^2$

$= 72 - 36 = 36$

$D > 0 \;\&\; f_{xx} > 0$

is $(6,18)$ a local min.

(17) Use Divergence Theorem to calculate the surface Integral $\iint_S F\, ds$ where

$F(x,y,z) = \langle x+y, y+z, x+z \rangle$

where S is sphere (center $(1,-2,4)$ and radius 10)

in other words the region in 3D space:

$\{(x,y,z) \mid (x-1)^2 + (y+2)^2 + (z-4)^2 = 100\}$