NAME: (print!) Prathik Lolla ________________________________

Section: ________ **E-Mail address:** ____________________________

MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)
Do not write below this line

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1. (out of 12)
2. (out of 12)
3. (out of 12)
4. (out of 12)
5. (out of 12)
6. (out of 12)
7. (out of 12)
8. (out of 12)
9. (out of 12)
10. (out of 12)
11. (out of 12)
12. (out of 12)
13. (out of 12)
14. (out of 12)
15. (out of 12)
16. (out of 12)
17. (out of 8)

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tot. (out of 200)

Important note: Unlike Exams 1 and 2, you are not required to state the type of the answer, and there is no credit for stating the type. But if the given answer is the **wrong type**, you would get 0 points.

Example: Find $f'(2)$ if $f(x) = x^3$. If you give the answer $3x^2$ instead of 12, you would get **zero** points!

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: 0

$x^2 + y^2 = 100$

$x = 10\cos t,\ y = 10\sin t$

$dx = -10\sin t\,dt,\ dy = 10\cos t\,dt$

$r(t) = \langle 10\cos t, 10\sin t \rangle$

$r'(t) = \langle -10\sin t, 10\cos t \rangle$

$|r'(t)| = \sqrt{100\sin^2 t + 100\cos^2 t} = 10$

$10\int_0^{2\pi} (70\sin t + 30\cos t)\,dt = 0$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $Z = 5x + 5y - 5$

$g(x,y,z) = x^2 + 3xy + y^2 - z$

$g_x = 2x + 3y, \; g_y = 3x + 2y, \; g_z = -1$

$g_x(1,1,5) = 5, \; g_y(1,1,5) = 5, \; g_z = -1$

$g_x(x - x_0) + g_y(y - y_0) + g_z(z - z_0) = 0$

$5(x-1) + 5(y-1) - (z-5) = 0$

$5x - 5 + 5y - 5 - z + 5 = 0$

$5x + 5y - z - 5 - \cancel{5} \cancel{+5} = 0$

$5x + 5y - z - 5 = 0$

$z = 5x + 5y - 5$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2\, y$ in the region

$$\{(x, y)\,|\,0 \le x \le 1\,,\, 0 \le y \le 1 - x\,\}.$$

Absolute minimum value: 0

Absolute maximum value: 0

$f(x,y) = x^2 y$

$f_x = 2xy\,,\ f_y = x^2$

$2xy = 0\,,\ x^2 = 0$

Critical Point @ $(0,1)$

$f(0,1) = 0$

$x = 0\,,\ (0,y)$

$f(0,y) = 0$ ← for y in $[0, 1-x]$

abs min: 0

abs max: 0

$y = 0\,,\ (x,0)$

$f(x,0) = 0$ ← for x in $[0,1]$

abs min: 0

abs max: 0

$y = 1-x\,,\ (x, 1-x)$

$f(x, 1-x) = x^2(1-x)$ ← for x in $[0,1]$

$g(x) = x^2(1-x) = x^2 - x^3$

$g'(x) = 2x - 3x^2$

$= 2x - 3x^2 = 0$

$g(0) = 0$

$g(1) = 0$

$g(\tfrac{2}{3}) = 0$

$g(0) = 0$

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(\,x^2\, +\, y\, +\, z\,) \quad .$$

Ans.: $f_{xxyz}(0,0,0) = -2$

$$f(x,y,z) = \sin(x^2+y+z)$$

$$f_x = 2x\cos(x^2+y+z)$$

$$f_{xx} = 2\cos(x^2+z+y) - 4x^2\sin(x^2+z+y)$$

$$f_{xxy} = -2(\sin(x^2+z+y) + 2x^2\cos(x^2+y+z))$$

$$f_{xxyz} = -2(2x^2\sin(z+y+x^2) + \cos(z+y+x^2))$$

$$f_{xxyz}(0,0,0) = -2(\cos(0)$$

$$f_{xxyz}(0,0,0) = -2$$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1,1,1)$ if (x,y,z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: $\frac{\partial z}{\partial y} = -1$ @ $(1,1,1)$

$F(x,y,z) = xy + xz + yz + x^2y^2z^2$

$F_y = x + z + 2x^2yz^2$

$F_z = x + y + 2x^2y^2z$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

F_y @ $(1,1,1)$

$F_y(1,1,1) = 1 + 1 + 2 = 4$

$F_z(1,1,1) = 1 + 1 + 2 = 4$

$$\frac{\partial z}{\partial y} = -\frac{4}{4}$$

$$\frac{\partial z}{\partial y} = -1$$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t\,,\; y = 2 + t\,,\; z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

and the line

$$x = -t\,,\; y = 1 + t\,,\; z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: $z = y - 1$

$n_1 = \langle 1, 1, 1 \rangle, \quad n_2 = \langle -1, 1, 1 \rangle$

$n_1 \times n_2 = \langle 0, -2, 2 \rangle$

$N = \langle 0, -2, 2 \rangle$

$N_1 (x - x_0) + N_2 (y - y_0) + N_3 (z - z_0)$

@ $\underline{t=0}$: $P = (1,2,3)$, $Q = (0, 1, 2)$

$-2(y-2) + 2(z-3) = 0$

$-2y + 4 + 2z - 6 = 0$

$-2y - 2 = -2z$

$y - 1 = z$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4 \sin 2t\,,\, -4 \cos 2t\,,\, 9e^{3t} \rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3 \rangle$ and its position at $t = 0$ is $\langle 0, 1, 1 \rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $x\left(\frac{\pi}{4}\right) = \langle 1 + \frac{\pi}{2}, 1, e^{\frac{3\pi}{4}} + \frac{3\pi}{4} + 1 \rangle$

$$V(t) = \int a(t)\,dt$$

$$= \int \langle -4\sin 2t, -4\cos 2t, 9e^{3t} \rangle\,dt$$

$$= \langle 2\cos(2t), -2\sin(2t), 3e^{3t} \rangle + \langle 2,0,3 \rangle$$

$$= \langle 2\cos(2t) + 2, -2\sin(2t), 3e^{3t} + 3 \rangle$$

$$x(t) = \int V(t)\,dt$$

$$= \int \langle 2\cos(2t) + 2, -2\sin(2t), 3e^{3t} + 3 \rangle\,dt$$

$$= \langle \sin(2t) + 2t, \cos(2t), e^{3t} + 3t \rangle + \langle 0,1,1 \rangle$$

$$= \langle \sin(2t) + 2t, \cos(2t) + 1, e^{3t} + 3t + 1 \rangle$$

$$x\left(\tfrac{\pi}{4}\right) = \langle \sin\left(\tfrac{\pi}{2}\right) + \tfrac{\pi}{2}, \cos\left(\tfrac{\pi}{2}\right) + 1, e^{\frac{3\pi}{4}} + \tfrac{3\pi}{4} + 1 \rangle$$

$$x\left(\tfrac{\pi}{4}\right) = \langle 1 + \tfrac{\pi}{2}, 1, e^{\frac{3\pi}{4}} + \tfrac{3\pi}{4} + 1 \rangle$$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2z)\, ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle t, 2t, 2t \rangle \quad , \quad 0 \leq t \leq 1 \quad .$$

Ans.: $\frac{7\sqrt{6}}{2}$

$f(x,y,z) = x + y + 2z$

$f(r(t)) = t + 2t + 4t = 7t$

$r'(t) = \langle 1, 1, 2 \rangle$

$|r'(t)| = \sqrt{1+1+4} = \sqrt{6}$

$\int_0^1 7\sqrt{6}\, t\, dt = \left[\frac{7\sqrt{6}}{2} t^2\right]_0^1 = \frac{7\sqrt{6}}{2}$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) \;=\; 1 \quad , \qquad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) \;=\; 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3} f(x,y,z)) \cos(\frac{\pi}{4} g(x,y,z)$$

Ans.: 0

$$\lim_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) = 0$$

10. (12 points) Compute

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\; y^2 + xz^3\,,\; z^2 + e^{xy}\,\rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \le x,y,z \le 1\}$$

with the normal pointing **outward**.

Ans.: 3

$$\mathbf{F} = \langle \underbrace{x^2 + \sin(y+z)}_{P}\,,\; \underbrace{y^2 + xz^3}_{Q}\,,\; \underbrace{z^2 + e^{xy}}_{R}\,\rangle$$

$P_x = 2x\,,\; Q_y = 2y\,,\; R_z = 2z$

$\operatorname{div} F = 2x + 2y + 2z$

$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$

$\int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz = 3$

11. (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x,y,z) = \langle 2e^{2x+3y+4z}, 3e^{2x+3y+4z}, 4e^{2x+3y+4z} \rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: $e^{12} - 1$

$$\mathbf{F}(x,y,z) = \langle \underset{P}{2e^{2x+3y+4z}}, \underset{Q}{3e^{2x+3y+4z}}, \underset{R}{4e^{2x+3y+4z}} \rangle$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix} = \langle 0,0,0 \rangle$$

F is conservative

$$\mathbf{F}(x,y,z) = \langle \underset{f_x}{2e^{2x+3y+4z}}, \underset{f_y}{3e^{2x+3y+4z}}, \underset{f_z}{4e^{2x+3y+4z}} \rangle$$

$f = \int f_x \, dx = e^{2x+3y+4z} + g(y,z)$

$f_y = 3e^{2x+3y+4z} + g_y \, , \; g_y = 0$

$f_z = 4e^{2x+3y+4z} + g_z \, , \; g_z = 0$

$f = e^{2x+3y+4z}$

$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2$

$r(t) = \langle t, 2t, t^2 \rangle$

$r(0) = (0,0,0) \; , \; r(1) = (1,2,1)$

$f(r(1)) - f(r(0))$

$f(1,2,1) - f(0,0,0) = e^{12} - 1$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\; y = t\,,\; z = t^2\,,\; 0 \le t \le 1.$

Ans.: 8

$x = t^2, \quad y = t, \quad z = t^2$

$dx = 2t\,dt, \quad dy = dt, \quad z = 2t\,dt$

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz$$

$$\int_0^1 5(t)(2t\,dt) + \int_0^1 5(t^2)(dt) + \int_0^1 6(t^2)(2t\,dt)$$

$$\int_0^1 10t^2\,dt + \int_0^1 5t^2\,dt + \int_0^1 12t^3\,dt$$

$$\left[\frac{10}{3}t^3\right]_0^1 + \left[\frac{5}{3}t^3\right]_0^1 + \left[3t^4\right]_0^1$$

$$\frac{10}{3} + \frac{5}{3} + \frac{9}{3} = \frac{24}{3}$$

$$8$$

13. (12 points) Evaluate

$$\int\int\int_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z)\,|\,x^2+y^2+z^2 \leq 100, z<0\} \quad .$$

Ans.: 100π

$\{(\rho,\theta,\phi) \mid 0 \leq \rho \leq 10,\ 0 \leq \theta \leq 2\pi,\ \frac{\pi}{2} \leq \phi \leq \pi\}$

$\int_0^{10}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\rho}\, \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$

$\left(\int_0^{10} \rho\, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_{\frac{\pi}{2}}^{\pi} \sin\phi\, d\phi\right) = 100\pi$

14. (12 points) Evaluate the quadruple integral

$$\int\int\int\int_E 360\,x\,dV \quad ,$$

where

$$E = \{(x,y,z,w)\,|\,0 \le w \le 1\,,\ 0 \le z \le w\,,\ 0 \le y \le z\,,\ 0 \le x \le y\} \quad .$$

Ans.: 15

$$\int_0^1 \int_0^w \int_0^z \int_0^y 360x \; dx\, dy\, dz\, dw$$

$$\int_0^y 360x\,dx = \left[180x^2\right]_0^y = 180y^2$$

$$\int_0^z 180y^2\,dy = \left[60y^3\right]_0^z = 60z^3$$

$$\int_0^w 60z^3\,dz = \left[15z^4\right]_0^w = 15w^4$$

$$\int_0^1 15w^4\,dw = \left[15w^4\right]_0^1 = 15$$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3

$$\begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix}$$

$X_u = 6\cos(2u + v)$ $\quad$ $Y_u = 1 - \sin(v + u)$

$X_v = 3\cos(2u + v)$ $\quad$ $Y_v = 1 - \sin(v + u)$

$X_u = 6$ $\quad$ $Y_u = 1$

$X_v = 3$ $\quad$ $Y_v = 1$

$J = X_u Y_v - X_v Y_u$

$J = 6 - 3$

$J = 3$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s): none

Local minimum points(s): (6, 18)

saddle point(s): (0,0)

$f(x,y) = x^3 + y^2 - 6xy$

$f_x = 3x^2 - 6y \quad f_y = 2y - 6x$

$f_{xx} = 6x \quad f_{yy} = 4$

$f_{xy} = -6$

Critical Points @ $(0, 0)$, $(6, 18)$

$(0,0) \Rightarrow$ saddle Point

$f_{xx} = 0$, $f_{yy} = 4$, $f_{xy} = -6$

$0(4) - [-6]^2 = -36 \quad D < 0$

$(6, 18) \Rightarrow$ local min

$f_{xx} = 36$, $f_{yy} = 4$, $f_{xy} = -6$

$36(4) - [-6]^2 = 108 \quad D > 0$

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\,\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = \langle\, x+y \;\, y+z \,,\, x+z \,\rangle \quad ,$$

where S is the sphere (center $(1,-2,4)$ and radius 10), in other words the region in 3D space:

$$\{(x,y,z)\,|\,(x-1)^2 \;+\; (y+2)^2 \;+\; (z-4)^2 = 100\} \quad .$$

$\mathbf{F}(x,y,z) = \langle\, \underset{P}{x+y} \;\, \underset{Q}{y+z} \,,\, \underset{R}{x+z} \,\rangle$

$\text{div } F = 1 + 1 + 1 = 3$

$\iint_S F \cdot dS = \iiint \text{div } F \, dV$

$V = \frac{4}{3}\pi (10)^3 (3) = 4000\pi$