NAME: (print!) Orion Kress-Sanfilippo

Section: 22 E-Mail address: omk23@scarlet..

MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)

Do not write below this line

---

1. (out of 12)
2. (out of 12)
3. (out of 12)
4. (out of 12)
5. (out of 12)
6. (out of 12)
7. (out of 12)
8. (out of 12)
9. (out of 12)
10. (out of 12)
11. (out of 12)
12. (out of 12)
13. (out of 12)
14. (out of 12)
15. (out of 12)
16. (out of 12)
17. (out of 8)

----

tot. (out of 200)

1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: Number: 400π

$X = 10\cos\theta \quad y = 10\sin\theta$ or $x = 10\cos(t) \quad y = 10\sin(t)$

$x'(t) = -10\sin(t) \quad y'(t) = 10\cos(t)$

Ans =

$$\int 7y \cdot x'(t) + 3x\, y'(t)\,dt$$

$$= \int_{2\pi}^{0} 7 \cdot 10\sin(t) \cdot -10\sin(t) + 300\cos^2(t)\,dt$$

$$= -\int_{0}^{2\pi} -700\sin^2(t) + 300\cos^2(t)\,dt =$$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: Eq of a plane: $5x + 5y - 5$

Check!

$5 \stackrel{?}{=} 1^2 + 3(1)(1) + 1^2 = 5$ ✓ pt $(1, 1, 5)$ is on existing plane

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

$$f_x = 2(1) + 3(1) + (0) = 5$$

$$f_y = 0 + 3(1) + 2(1) = 5$$

$$z - 5 = 5(x-1) + 5(y-1)$$

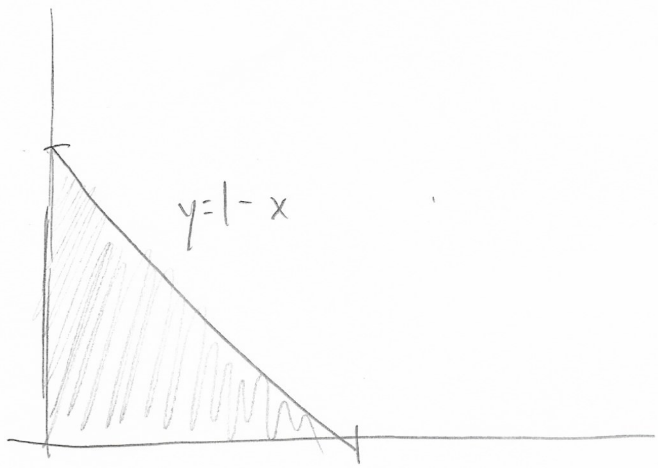
$$\boxed{z = 5x + 5y - 5}$$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2\, y$ in the region

$$\{(x, y) \mid 0 \le x \le 1\,,\; 0 \le y \le 1 - x\,\}.$$

Absolute minimum value: 0

Absolute maximum value: 4/27



$f(x) = x^2(1-x)$ $0 \le x \le 1$

$f'(x) = 2x - 3x^2 = 0$

$x = 0$

$2 - 3x = 0$

$\boxed{x = \frac{2}{3}}$

$f\left(\frac{2}{3}, 1 - \frac{2}{3}\right)$

$= \left(\frac{2}{3}\right)^2\left(1 - \frac{2}{3}\right)$

$= \frac{4}{9} \cdot \frac{1}{3} = \boxed{\frac{4}{27}}$

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(\,x^2\, +\, y\, +\, z\,) \quad .$$

Ans.: Number : -2

$$f_x = \cos(x^2+y+z)\cdot 2x$$

$$f_{xx} = \left(\left(-\sin(x^2+y+z)\cdot 2x\right)\cdot 2x + \cos(x^2+y+z)\cdot 2\right)$$

$$f_{xxy} = -\cos(x^2+y+z)\cdot 4x^2 - \sin(x^2+y+z)\cdot 2$$

$$f_{xxyz} = \sin(x^2+y+z)\cdot 4x^2 - \cos(x^2+y+z)\cdot 2$$

$$f_{xxyz}(0,0,0) = \sin(0)\cdot 0 - \cos(0)\cdot 2$$

$$\boxed{= -2}$$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1, 1, 1)$ if (x, y, z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: Number: $\frac{-3}{4}$

$(xy + xz + yz + x^2y^2z^2 = 4)'$ diff $z(y)$ w.r.t. y

$x + xz' + z' + x^2(2yz^2 + 2y^2zz') = 0$

$z'(x + 1 + 2x^2y^2z) + x + 2x^2yz^2 = 0$

$\frac{\partial z}{\partial y} = \frac{-x - 2x^2yz^2}{x + 1 + 2x^2y^2z}$ @ $(1,1,1) =$

$= \frac{-1 - 2(1)(1)(1)}{1 + 1 + 2(1)(1)(1)} = \boxed{\frac{-3}{4}}$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t\,,\ y = 2 + t\,,\ z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

and the line

$$x = -t\,,\ y = 1 + t\,,\ z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: Eq for a plane: $-y + z = 1$

$r_1(t) = (1, 2, 3) + t\langle 1, 1, 1\rangle$ PQ

$r_2(t) = (0, 1, 2) + t\langle -1, 1, 1\rangle$ PR

$$PQ \times PR = \det \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (0)\hat{i} - (1-(-1))\hat{j} + (1-(-1))\hat{k}$$

$$= \langle 0, -2, 2\rangle$$

$$0(x-1) + (-2)(y-2) + 2(z-3) = 0$$

$$\boxed{-y + z = 1}$$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4\sin 2t\,,\ -4\cos 2t\,,\ 9e^{3t}\,\rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3\rangle$ and its position at $t = 0$ is $\langle 0, 1, 1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, e^{\frac{3\pi}{4}}\right)$ Point

$$\int a(t)\,dt = v(t) = \int \langle -4\sin 2t, -4\cos(2t), 9e^{3t}\rangle \cdot dt$$

$$= \langle 2\cos(2t), -2\sin(2t), 3e^{3t}\rangle + \vec{C}_1$$

$$@\,(t=0) \quad 2\cos(2(0)) + C_1, -2\sin(2(0)), 3e^{0} = (2, 0, 3)$$

$$2(1) \qquad 0 \qquad 3(1) \qquad \boxed{\vec{C}_1 = 0}$$

$$@\,t=0$$

$$\int v(t)\,dt = \boxed{x(t) = \sin(2t), \cos(2t), e^{3t}} = (0, 1, 1)$$

$$0 \qquad 1 \qquad 1$$

$$x\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, e^{\frac{3\pi}{4}}$$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2\,z)\, ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle\, t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: Number: $\frac{21}{2}$

$r'(t) = \langle 1, 2, 2 \rangle$

$|r'(t)| = \sqrt{1 + 4 + 4} = 3$

$\int_0^1 (t + 2t + 2(2t)) \cdot 3\, dt$

$= \frac{21}{2}$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3} f(x,y,z)\right) \cos\left(\frac{\pi}{4} g(x,y,z)\right)$$

Ans.: 0

$$= \sin\left(\frac{\pi}{3}(1)\right)\cos\left(\frac{\pi}{4}\cdot(2)\right)$$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$$

$$= \boxed{0}$$

10. (12 points) Compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\, y^2 + xz^3\,,\, z^2 + e^{xy} \rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x, y, z) \,|\, 0 \leq x, y, z \leq 1\}$$

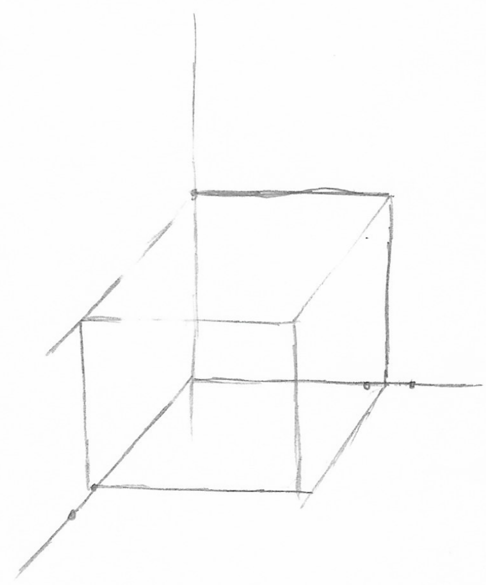
with the normal pointing **outward**.

Ans.: Number: 3

Divergence Thm

$$\iint_S F \cdot d\underline{S} = \iiint_E \mathrm{div}(F)\, dV$$

$$\mathrm{div}(F) = 2x + 2y + 2z$$



Ans= $$\int_0^1\int_0^1\int_0^1 2x + 2y + 2z \; dx\,dy\,dz$$

$$= \boxed{3}$$

11. (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) = \langle 2e^{2x+3y+4z}, 3e^{2x+3y+4z}, 4e^{2x+3y+4z} \rangle ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \ .$$

Ans: Number: $e^{12} - 1$

Check Conservative:

$$\det \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2\exp(2x+3y+4z) & 3\exp(2x+3y+4z) & " \end{vmatrix}$$

$= (12\exp(\ldots) - 12\exp(\ldots))\hat{i}$

$- (8\exp(\ldots) - 8\exp(\ldots))\hat{j}$

$+ (6\exp(\ldots) - 6\exp(\ldots))\hat{k}$

$= 0$ ✓

$f = e^{2x+3y+4z}$

By fundamental thm of line integrals: if $F = \nabla f$

$\int F \cdot d\underline{r} = f(b) - f(a)$

$= e^{2(t)+3(2t)+4(t^2)} \Big|_0^1$

$= e^{(2+6+4)} - 1$

$= \boxed{e^{12} - 1}$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\ y = t\,,\ z = t^2\,,\ 0 \le t \le 1.$

Ans.: Number : 8

$$r'(t)dt = \langle 2t, 1, 2t\rangle$$

$$\int_0^1 5(t)\cdot 2t + 5(t^2)\cdot 1 + 6(t^2)\cdot 2t\,dt$$

$$= \int_0^1 12t^3 + 15t^2\,dt$$

$$= 8$$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z) \,|\, x^2+y^2+z^2 \leq 100, z<0\} \quad .$$

Ans.: Number : 100π

$x^2+y^2+z^2=\rho^2 \qquad dV=\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$E: \{(\rho, \theta, \phi) \,|\, 0 \leq \rho \leq 10,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{2}\}$

$$\text{Ans} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{10} \frac{1}{\rho}\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= 100\pi$$

14. (12 points) Evaluate the quadruple integral

$$\int\int\int\int_E 360\,x\,dV \quad ,$$

where

$$E = \{(x,y,z,w)\,|\,0 \le w \le 1\,,\,0 \le z \le w\,,\,0 \le y \le z\,,\,0 \le x \le y\} \quad .$$

Ans.: Number: $\boxed{3}$

$$= \int_0^1\int_0^w\int_0^z\int_0^y 360x\,dx\,dy\,dz\,dw$$

$$= \int_0^1\int_0^w\int_0^z 180y^2\,dy\,dz\,dw = \int_0^1\int_0^w 60z^3\,dz\,dw$$

$$= \int_0^1 15w^4\,dw = \boxed{3}$$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: Number: 3

$X_u = 6\cos(2u+v)$ $\quad 1-\sin(u+v) = y_u$

$X_v = 3\cos(2u+v)$ $\quad 1-\sin(u+v) = y_v$

$Jac(u,v) = 6\cos(2u+v)(1-\sin(u+v)) - 3\cos(2u+v)(1-\sin(u+v))$

$= 3\cos(2u+v)(1-\sin(u+v))$

$@(0,0) = 3\cos(0)(1-\sin(0))$

$= 3(1)(1) = \boxed{3}$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s):

Local minimum points(s): $(6, 18)$

saddle point(s): $(0, 0)$

$f_x = 3x^2 - 6y \quad f_y = 2y - 6x$

$f_{xx} = 6x \quad f_{yy} = 2$

$f_{xy} = -6$

$f_x = 0 = 3x^2 - 6y = 3x^2 - 6(3x) = 0$

$f_y = 0 = 2y - 6x \Rightarrow \boxed{x = 0} \; \boxed{y = 0}$

$3x - 18 = 0$

$y = 3x \quad \boxed{x = 6} \; \boxed{y = 18}$

Discriminant

$f_{xx} f_{yy} - [f_{xy}]^2 = 12x - (36)$

@ $(0, 0)$: $12(0) - 36 < 0$ Saddle Pt.

@ $(6, 18)$: $12(6) - 36 > 0$ $\therefore$ Local

& $f_{xx} > 0$ Min

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle\, x + y, y + z \,,\, x + z \,\rangle \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z) \,|\, (x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 100\} \quad .$$

$$\iint_S F \cdot d\underline{S} = \iiint_E \operatorname{div}(F)\, dV$$

$$\operatorname{div}(F) = 3$$

Special Case of Volume Int:

Integrand is $\underline{\text{const}}$,

Integral is $\underline{\text{Volume}} \cdot \underline{\text{Const}}$

$$= 3 \cdot \frac{4}{3} \pi r^3 = 4\pi(1000)$$

$$= \boxed{4000\pi}$$