NAME: (print!) Irina Mukhametzhanova

Section: 24 **E-Mail address:** iim15@scarletmail.rutgers.edu

MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)

Do not write below this line

---

1. (out of 12)
2. (out of 12)
3. (out of 12)
4. (out of 12)
5. (out of 12)
6. (out of 12)
7. (out of 12)
8. (out of 12)
9. (out of 12)
10. (out of 12)
11. (out of 12)
12. (out of 12)
13. (out of 12)
14. (out of 12)
15. (out of 12)
16. (out of 12)
17. (out of 8)

----

tot. (out of 200)

Important note: Unlike Exams 1 and 2, you are not required to state the type of the answer, and there is no credit for stating the type. But if the given answer is the **wrong type**, you would get 0 points.

Example: Find $f'(2)$ if $f(x) = x^3$. If you give the answer $3x^2$ instead of 12, you would get **zero** points!

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: -400π

We can use Green's Theorem to solve the integral. If we have a vector field $\langle P, Q \rangle$, we can rewrite the integral as:

$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

First, find Q_x and P_y of our vector field:

$Q = 3x \rightarrow Q_x = 3$

$P = 7y \rightarrow P_y = 7$

So, our new integral is:

$\iint_D 3 - 7\,dA = \iint_D -4\,dA$

The area integral of a constant is the area of the region times that constant. We have a circle of radius 10, so, it area is:

$\pi r^2 = \pi (10)^2 = 100\pi$

So, the value of the line integral is:

$-4 \cdot 100\pi = -400\pi$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $5x + 5y - z = 5$

First, check that the point $(1,1,5)$ makes sense:

$5 = (1)^2 + 3(1)(1) + (1)^2 = 1+3+1 = 5$ ✓

The equation of a plane with a normal vector $\langle n_1, n_2, n_3 \rangle$ at point (x_0, y_0, z_0) is:

$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$

We can find the normal vector by taking the gradient of the function:

$\nabla f = \langle f_x, f_y \rangle = \langle 2x+3y, 3x+2y \rangle$

$\nabla f(1,1) = \langle 5, 5 \rangle$

The equation of our plane is:

$5(x-1) + 5(y-1) = z - 5$

$5x - 5 + 5y - 5 = z - 5$

$\boxed{5x + 5y - z = 5}$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2 y$ in the region

$$\{(x, y) \,|\, 0 \leq x \leq 1\,,\, 0 \leq y \leq 1 - x\,\}.$$

Absolute minimum value: 0

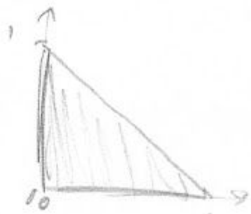
Absolute maximum value: $\frac{4}{27}$

First, we need to find critical point (or possible locations of minima and maxima) by setting f_x and f_y to 0.

$f_x = 2xy = 0$

$f_y = x^2 = 0 \Rightarrow x = 0 \rightarrow y =$ any real number

Region:



We have 3 lines to check: $x=0$, $y=0$, and $y=1-x$

$x=0$: $f(0,y) = 0$

$y=0$: $f(x,0) = 0$

$y=1-x$: $f(x,1-x) = x^2(1-x) = x^2 - x^3 \Rightarrow$

$\rightarrow f(x,1-x)_x = 2x - 3x^2 = 0 \Rightarrow 2x = 3x^2 \Rightarrow 2 = 3x \Rightarrow x = 0, \frac{2}{3}$

At $x=0$, $f(x,1-x)(0) = 0$

At $x = \frac{2}{3}$, $f(x,1-x)(0) = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}$

At $x = 1$, $f(x,1-x) = 1-1 = 0$

$\boxed{\text{Absolute min} = 0, \text{ absolute max} = \frac{4}{27}}$

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(\,x^2\, +\, y\, +\, z\,) \quad .$$

Ans.: -2

$\frac{\partial}{\partial z} \sin(x^2+y+z) = \cos(x^2+y+z)$

$\frac{\partial}{\partial y} \cos(x^2+y+z) = -\sin(x^2+y+z)$

$\frac{\partial}{\partial x} -\sin(x^2+y+z) = -2x\cos(x^2+y+z)$

$\frac{\partial}{\partial x} -2x\cos(x^2+y+z) = (2x)(2x\sin(x^2+y+z)) + (-2)(\cos(x^2+y+z)) =$

$= f_{xxyz}(0,0,0) = (2\cdot 0)(2\cdot 0\cdot \sin(0)) - 2(\cos(0+0+0)) = 0 - 2\cos(0) = -2$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1,1,1)$ if (x,y,z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: -1

First, check if the point makes sense:

$(1)(1) + (1)(1) + (1)(1) + (1)^2(1)^2(1)^2 = 1+1+1+1 = 4 = 4$ ✓

Treat z as a function $z(x,y)$. When we take the derivative w.r.t. y, x is treated as a constant. So, take that derivative of both sides:

$(xy)' + (xz)' + (yz)' + (x^2y^2z^2)' = 0$

$x + xz' + yz' + z + (x^2y^2)(2zz') + (2x^2y)(z^2) = 0$

$x + xz' + yz' + z + 2x^2y^2zz' + 2x^2yz^2 = 0$

Single and factor out z':

$xz' + yz' + 2x^2y^2zz' = -x - z - 2x^2yz^2$

$z' = \dfrac{-x-z-2x^2yz^2}{x+y+2x^2y^2z}$

Plug in the point $(1,1,1)$:

$z'(1,1,1) = \dfrac{-1-1-2(1)^2(1)(1)^2}{1+1+2(1)^2(1)^2(1)} = \dfrac{-4}{4} = \boxed{-1}$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t,\, y = 2 + t,\, z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

and the line

$$x = -t,\, y = 1 + t,\, z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: $-y + z = 1$

The first line is parallel to the vector $\langle 1, 1, 1 \rangle$, and the second line is parallel to the vector $\langle -1, 1, 1 \rangle$.

We can find a vector perpendicular to both of the vectors above, which would be our plane's normal vector:

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \hat{i}(1-1) - \hat{j}(1+1) + \hat{k}(1+1) = \langle 0, -2, 2 \rangle$$

The formula for a plane with normal vector $\langle n_1, n_2, n_3 \rangle$ and with a point (x_0, y_0, z_0) is:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

At $t = 0$, line #1 is at a point $(1+0, 2+0, 3+0) = (1, 2, 3)$

So, the equation of our plane is:

$$0(x-1) - 2(y-2) + 2(z-3) = 0$$

$$-2y + 4 + 2z - 6 = 0$$

$$-2y + 2z = 2$$

$$-y + z = 1$$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4\sin 2t\,, -4\cos 2t\,, 9e^{3t} \rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3\rangle$ and its position at $t = 0$ is $\langle 0, 1, 1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $(1, 0, e^{\frac{3\pi}{4}})$

Find the velocity vector by integrating the acceleration vector:

$\int a(t)\,dt = \langle \int -4\sin(2t)\,dt, \int -4\cos(2t)\,dt, \int 9e^{3t}\,dt \rangle =$

$= \langle 2\cos(2t), -2\sin(2t), 3e^{3t} \rangle + C$, where C is an arbitrary vector.

Use the fact that $v(0) = \langle 2, 0, 3\rangle$ to find C:

$\langle 2\cos(2\cdot 0), -2\sin(2\cdot 0), 3e^{3\cdot 0}\rangle + C = \langle 2, 0, 3\rangle$

$\langle 2, 0, 3\rangle + C = \langle 2, 0, 3\rangle \Rightarrow C = \langle 0, 0, 0\rangle$

So, $v(t) = \langle 2\cos(2t), -2\sin(2t), 3e^{3t}\rangle$

Find the position vector by integrating the velocity vector:

$\int v(t)\,dt = \langle \int 2\cos(2t)\,dt, \int -2\sin(2t)\,dt, \int 3e^{3t}\,dt \rangle =$

$= \langle \sin(2t), \cos(2t), e^{3t}\rangle + C$, where C is an arbitrary vector.

Use the fact that $r(0) = \langle 0, 1, 1\rangle$ to find C:

$\langle \sin(2\cdot 0), \cos(2\cdot 0), e^{3\cdot 0}\rangle + C = \langle 0, 1, 1\rangle$

$\langle 0, 1, 1\rangle + C = \langle 0, 1, 1\rangle \Rightarrow C = \langle 0, 0, 0\rangle$

So, our position vector is:

$r(t) = \langle \sin(2t), \cos(2t), e^{3t}\rangle$

Plug in $t = \frac{\pi}{4}$

$r(\frac{\pi}{4}) = \langle \sin(\frac{\pi}{2}), \cos(\frac{\pi}{2}), e^{\frac{3\pi}{4}}\rangle = \boxed{\langle 1, 0, e^{\frac{3\pi}{4}}\rangle}$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2z)\, ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: $\frac{21}{2}$

We can first find ds using the formula:

$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

Now, in the integral, substitute $x=t$, $y=2t$, $z=2t$ (from the parametric equations), and $ds=3$ (from results above):

$\int_C x+y+2z\, ds = \int_0^1 (t+2t+4t)\,3\,dt = 3\int_0^1 7t\,dt =$

$= 3\left(\frac{7}{2}t^2\right)\Big|_0^1 = \boxed{\frac{21}{2}}$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3} f(x,y,z)) \cos(\frac{\pi}{4} g(x,y,z))$$

Ans.: 0

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\tfrac{\pi}{3} f(x,y,z)) \cos(\tfrac{\pi}{4}(g(x,y,z))) =$$

$$= \lim_{(x,y,z)\to(1,1,1)} \sin(\tfrac{\pi}{3} f(x,y,z)) \cdot \lim_{(x,y,z)\to(1,1,1)} \cos(\tfrac{\pi}{4}(g(x,y,z)))$$

$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$, if f is continuous at $\lim_{x\to a} g(x)$

$$\frac{\pi}{3} \cdot \lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = \frac{\pi}{3} \cdot 1 = \frac{\pi}{3} \Rightarrow \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

$$\frac{\pi}{4} \cdot \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = \frac{2\pi}{4} = \frac{\pi}{2} \Rightarrow \cos(\frac{\pi}{2}) = 0$$

$$\frac{\sqrt{3}}{2} \cdot 0 = \boxed{0}$$

10. (12 points) Compute

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\, y^2 + xz^3\,,\, z^2 + e^{xy} \rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \le x,y,z \le 1\}$$

with the normal pointing **outward**.

Ans.: 3

We can use the Divergence Theorem, which states that:

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\,dV$

First, we need to find divergence:

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2+\sin(y+z)) + \frac{\partial}{\partial y}(y^2+xz^3) + \frac{\partial}{\partial z}(z^2+e^{xy}) =$

$= 2x + 2y + 2z$

Our region is in the question: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

Our final integral is:

$\int_0^1\int_0^1\int_0^1 2x+2y+2z\,dx\,dy\,dz = \int_0^1\int_0^1 x^2 + 2yx + 2zx\Big|_0^1 dy\,dz =$

$= \int_0^1\int_0^1 1+2y+2z\,dy\,dz = \int_0^1 y+y^2+2zy\Big|_0^1 dz = \int_0^1 1+1+2z\,dz =$

$= \int_0^1 2+2z\,dz = 2z+z^2\Big|_0^1 = 2+1 = \boxed{3}$

11. (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) = \langle 2e^{2x+3y+4z}, 3e^{2x+3y+4z}, 4e^{2x+3y+4z} \rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: $e^{12} - 1$

To check if we can find f, we need to see if $\vec{F}$ is conservative, or path independent, by seeing if curl$(\vec{F}) = \vec{0}$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2e^{2x+3y+4z} & 3e^{2x+3y+4z} & 4e^{2x+3y+4z} \end{vmatrix} =$$

$= \hat{\imath}\,(12e^{2x+3y+4z} - 12e^{2x+3y+4z}) - \hat{\jmath}\,(8e^{2x+3y+4z} - 8e^{2x+3y+4z}) + \hat{k}\,(6e^{2x+3y+4z} - 6e^{2x+3y+4z}) = \vec{0}$ ✓

Because $F[1] = f_x$, we can integrate $F[1]$ w.r.t. x:

$\int 2e^{2x+3y+4z}\,dx = e^{2x+3y+4z} + g(y,z)$

Take the derivative w.r.t. y of the result to later find $g(y,z)$:

$3e^{2x+3y+4z} + g_y = 3e^{2x+3y+4z} \Rightarrow g_y = 0 \Rightarrow g(y,z) = 0 \Rightarrow$

$\rightarrow f(x,y,z) = e^{2x+3y+4z} + h(z)$

Take the derivative w.r.t. z of the result to later find $h(z)$:

$4e^{2x+3y+4z} + h_z = 4e^{2x+3y+4z} \Rightarrow h_z = 0 \Rightarrow h(z) = 0 \Rightarrow$

$\rightarrow f(x,y,z) = e^{2x+3y+4z}$

Because $\vec{F}$ is conservative, the path in between a and b does not matter. So:

$\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start})$

Start: $r(0) = (0,0,0)$ End: $r(1) = (1,2,1)$

$f(1,2,1) - f(0,0,0) = e^{2+6+4} - e^0 = \boxed{e^{12} - 1}$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\ y = t\,,\ z = t^2\,,\ 0 \le t \le 1.$

Ans.: 8

First, find dx, dy, and dz by finding $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$, $\frac{\partial z}{\partial t}$ and multiplying those by dt (use the curve):

$dx = 2t\,dt \quad dy = dt \quad dz = 2t\,dt$

Convert the vector field to a function of t using the curve's equations:

$5y = 5t \quad 5x = 5t^2 \quad 6z = 6t^2$

Using the t-bounds in the question, our final integral is:

$\int_0^1 5t(2t) + 5t^2 + 6t^2(2t)\,dt = \int_0^1 10t^2 + 5t^2 + 12t^3\,dt =$

$= \int_0^1 15t^2 + 12t^3\,dt = 5t^3 + 3t^4 \Big|_0^1 = 5 + 3 = \boxed{8}$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z)\,|\,x^2+y^2+z^2 \le 100, z<0\} \quad .$$

Ans.: 100π

We can see that one of the bounds is a sphere of radius 10.
So, we can use spherical coordinates:

$x = r\sin\phi\cos\theta \quad y = r\sin\phi\sin\theta \quad z = r\cos\phi \quad r = \sqrt{x^2+y^2+z^2}$

$dV = r^2\sin\phi \; dr\,d\phi\,d\theta$

With new bounds and these conversions, our new integral is:

$\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\int_0^{10} \frac{1}{r} r^2\sin\phi \, dr\,d\theta\,d\phi = \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\int_0^{10} r\sin\phi\,dr\,d\theta\,d\phi =$

$= \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \frac{r^2}{2}\sin\phi \Big|_0^{10} d\theta\,d\phi = \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} 50\sin\phi\,d\theta\,d\phi = \int_{\frac{\pi}{2}}^{\pi} 50\theta\sin\phi\Big|_0^{2\pi} d\phi =$

$= \int_{\frac{\pi}{2}}^{\pi} 100\pi\sin\phi\,d\phi = -100\pi\cos\phi\Big|_{\frac{\pi}{2}}^{\pi} = -100\pi\cos(\pi) + 100\pi\cos(\frac{\pi}{2}) = 100\pi$

14. (12 points) Evaluate the quadruple integral

$$\iiiint_E 360\, x\, dV \quad ,$$

where

$$E = \{(x, y, z, w) \,|\, 0 \le w \le 1\,,\ 0 \le z \le w\,,\ 0 \le y \le z\,,\ 0 \le x \le y\} \quad .$$

Ans.: 3

$$\int_0^1\int_0^w\int_0^z\int_0^y 360x\,dx\,dy\,dz\,dw = \int_0^1\int_0^w\int_0^z 180x^2\Big|_0^y dy\,dz\,dw =$$

$$= \int_0^1\int_0^w\int_0^z 180y^2\,dy\,dz\,dw = \int_0^1\int_0^w 60y^3\Big|_0^z dz\,dw =$$

$$= \int_0^1\int_0^w 60z^3\,dz\,dw = \int_0^1 15z^4\Big|_0^w dw = \int_0^1 15w^4\,dw =$$

$$= 3w^5\Big|_0^1 = \boxed{3}$$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3

The formula for the Jacobian is:

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

So, find the value of each partial derivative at $(0,0)$:

$x_u = 2\cdot 3\cos(2u+v) = 6\cos(2u+v) \rightarrow x_u(0,0) = 6\cos(0) = 6$

$x_v = 3\cos(2u+v) \rightarrow x_v(0,0) = 3\cos(0) = 3$

$y_u = 1 - \sin(u+v) \rightarrow y_u(0,0) = 1 - \sin(0) = 1$

$y_v = 1 - \sin(u+v) \rightarrow y_v(0,0) = 1 - \sin(0) = 1$

So, our Jacobian is:

$$J(0,0) = \begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix} = 6 - 3 = \boxed{3}$$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s): None

Local minimum points(s): $(6, 18)$

saddle point(s): $(0, 0)$

First, we need to find critical points, or where max/min/saddle points could possibly be located. These points occur where $f_x = f_y = 0$.

$f_x = 3x^2 - 6y = 0 \Rightarrow 3x^2 = 6y \rightarrow x^2 = 2y$

$f_y = 2y - 6x = 0 \rightarrow x^2 - 6x = 0 \rightarrow x = 0, 6 \rightarrow y = 6, 18$

So, our critical points are $(0,0)$ and $(6, 18)$

To determine whether each point is a max/min/saddle point, we can use the second derivative test:

$D = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}(x,y)^2$

If $D > 0$ and $f_{xx}(x,y) > 0$, it is a local minimum.

If $D > 0$ and $f_{xx}(x,y) < 0$, it is a local maximum.

If $D < 0$, it is a saddle point.

If $D = 0$, it is inconclusive.

$f_{xx} = 6x \quad f_{xy} = -6 \quad f_{yy} = 2$

$D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2 = (0)(2) - 36 = -36$

Because $D < 0$, $\boxed{(0,0) \text{ is a saddle point}}$

$D(6,72) = f_{xx}(6,18) f_{yy}(6,18) - f_{xy}(6,18)^2 = (36)(2) - 36 = 36$

Because $D > 0$, and $f_{xx}(6,72) > 0$, $\boxed{(6,18) \text{ is a local minimum}}$

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle x + y \; y + z \, , \, x + z \rangle \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z) \,|\, (x-1)^2 + (y+2)^2 + (z-4)^2 = 100\} \quad .$$

The divergence theorem states that:

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV$

First, find $\operatorname{div}(\vec{F})$:

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+z) = 1+1+1 = 3$

So, our integral is:

$\iiint_E 3 \, dV$

The volume integral of a constant is the volume of the region times that constant. We have a sphere of radius 10, so, its volume is:

$\frac{4}{3}\pi(10)^3 = \frac{4}{3}\pi(1000) = \frac{4000\pi}{3}$

Times the constant:

$3 \cdot \frac{4000\pi}{3} = \boxed{4000\pi}$