1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

$r = 10$

Ans.: 400π

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multiply by -1 for clockwise

$$\iint_D 3 - 7 \, dA = \iint_D -4 \, dA$$

$$\int_0^{2\pi}\int_0^{10} -4r \, dr \, d\theta$$

$$\int_0^{2\pi} \frac{-4r^2}{2}\Big|_0^{10} d\theta = \int_0^{2\pi} -200 \, d\theta$$

$$-200(2\pi - 0) = -400\pi$$

$$-400\pi \cdot -1 = 400\pi$$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $z = 5x - 5y - 5$

$5 = 1^2 + 3(1)(1) + (1^2)$

$5 = 1 + 3 + 1$

$5 = 5$ ✓

$f_x = 2x + 3y \qquad f_x(1,1,5) = 2(1) + 3(1) = 5$

$f_y = 3x + 2y \qquad f_y(1,1,5) = 3(1) + 2(1) = 5$

$z - 5 = 5(x-1) + 5(y-1)$

$z = 5x - 5 + 5y - 5 + 5$

$z = 5x - 5y - 5$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2\, y$ in the region

$0 \le y \le 1$

$$\{(x, y) \,|\, 0 \le x \le 1,\, 0 \le y \le 1 - x\}.$$

$1 - 0 = 1$
$1 - 1 = 0$

Absolute minimum value: 0

Absolute maximum value: 0 no absolute max or min

$f_x = 2xy$ $\quad$ $f_y = x^2$ $\quad$ $f(0,0) = 0$

$0 = 2xy$ $\quad$ $0 = x^2$ $\quad$ $f(0,1) = 0$

$0 = 2(0)y$ $\quad$ $x = 0$ $\quad$ $f(1,0) = 0$

$y = 0$

$f_{xx} \cdot f_{yy} - (f_{xy})^2$

$2y \cdot 0 - (2x)^2$

$-4x^2$

$D(0,0) = -4(0) = 0$ test fails

$D(1,0) = -4(1^2) = -4$ no min or max

$D(0,1) = -4(0) = 0$ test fails

$f(0,0) = 0$

$f(1,0) = 0$

$f(0,1) = 0$

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(x^2 + y + z) \quad .$$

Ans.: -2

$f_x = \cos(x^2+y+z) \cdot 2x$

$f_{xx} = -\sin(x^2+y+z) \cdot 2x \cdot 2x + 2 \cdot \cos(x^2+y+z)$

$f_{xxy} = -\cos(x^2+y+z) \cdot 2x + 2 - \sin(x^2+y+z) \cdot 1$

$f_{xxyz} = \sin(x^2+y+z) \cdot 2x \cdot 1 - 2\cos(x^2+y^2+z) \cdot 1$

$f_{xxyz}(0,0,0) = \sin(0) \cdot 2(0) - 2\cos(0)$

$= -2$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1, 1, 1)$ if (x, y, z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: $\frac{-3}{4}$

x is 0

$$\frac{d}{dy}(xy + xz + yz + x^2y^2z^2) = \frac{d}{dy}(4)$$

$$\frac{dz}{dy} \cdot x + z + \frac{dz}{dy} \cdot y + y^2\left(2z \cdot \frac{dz}{dy} \cdot y^2 + 2y \cdot z^2\right) = 0$$

(1,1,1)

$$\frac{dz}{dy} \cdot 1 + 1 + \frac{dz}{dy} \cdot 1 + 1^2\left(2(1) \cdot \frac{dz}{dy} \cdot 1^2 + 2(1) \cdot 1^2\right) = 0$$

$$2\frac{dz}{dy} + 1 + 2\frac{dz}{dy} + 2 = 0$$

$$4\frac{dz}{dy} = -3$$

$$\frac{dz}{dy} = \frac{-3}{4}$$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t,\ y = 2 + t,\ z = 3 + t \quad (-\infty < t < \infty) \ ,$$

and the line

$$x = -t,\ y = 1 + t,\ z = 2 + t \quad (-\infty < t < \infty) \ .$$

Ans.: $z = y + 2$

$L_1 = \langle 1, 2, 3 \rangle + t \langle 1, 1, 1 \rangle$

$L_2 = \langle 0, 1, 2 \rangle + t \langle -1, 1, 1 \rangle$

need normal to plane and point → any random t

$L_1(0) = \langle 1, 2, 3 \rangle + 0 \langle 1, 1, 1 \rangle$

$L_1(0) = \langle 1, 2, 3 \rangle$

$$L_1 \times L_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0i - (1+1)j + (1+1)k$$

$= -2j + 2k$

$0(x-1) - 2(y-2) + 2(z-3) = 0$

$-2y + 4 + 2z - 6 = 0$

$y + 2 = z$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4 \sin 2t\,,\ -4 \cos 2t\,,\ 9e^{3t} \rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3 \rangle$ and its position at $t = 0$ is $\langle 0, 1, 1 \rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $\langle 1, 0, e^{3\pi/4} \rangle$

$a(t) = \frac{d}{dt} v(t)$

$u = 2t$

$v(t) = \langle 4 \cos 2t \cdot \frac{1}{2}, -4 \sin 2t \cdot \frac{1}{2}, 3e^{3t} \rangle + C$

$u = 2t$

$du = 2dt$

$\frac{1}{2} du = dt$

$v(2,0,3) = \langle 2\cos 0, -2\sin 0, 3e^{0} \rangle + C$

$\langle 2, 0, 3 \rangle = \langle 2, 0, 3 \rangle + C$

$C = 0$

$v(t) = \langle 2\cos 2t, -2\sin 2t, 3e^{3t} \rangle$

$x(t) = \langle \sin 2t, \cos 2t, e^{3t} \rangle + C$

$\langle 0, 1, 1 \rangle = \langle \sin 0, \cos(0), e^{0} \rangle + C$

$\langle 0, 1, 1 \rangle = \langle 0, 1, 1 \rangle + C$

$C = 0$

$x(\frac{\pi}{4}) = \langle \sin 2 \cdot \frac{\pi}{4}, \cos 2 \cdot \frac{\pi}{4}, e^{\frac{3\pi}{4}} \rangle = \langle 1, 0, e^{3\pi/4} \rangle$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2\,z)\, ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: 7

$|r'(t)| = \sqrt{t^2 + 4t^2 + 4t^2}$

$|r'(t)| = \sqrt{9t^2}$

$|r'(t)| = 3t$

$\int_0^1 (t + 2t + 4t) \cdot 3t \, dt$

$3\int_0^1 t^2 + 2t^2 + 4t^2 \, dt$

$3\left(\frac{t^3}{3} + \frac{2t^3}{3} + \frac{4t^3}{3}\Big|_0^1\right) = \left(\frac{1}{3} + \frac{2}{3} + \frac{4}{3}\right)3 = 7$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3} f(x,y,z)) \cos(\frac{\pi}{4} g(x,y,z))$$

Ans.: 0

$$\lim_{(x,y,z)\to(1,1,1)} \sin \frac{\pi}{3}(1) \cdot \cos(\frac{\pi}{4} \cdot 2)$$

$$\lim_{(x,y,z)\to(1,1,1)} = 0$$

10. (12 points) Compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = (x^2 + \sin(y+z)\,,\, y^2 + xz^3\,,\, z^2 + e^{xy})$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x, y, z) \,|\, 0 \le x, y, z \le 1\}$$

with the normal pointing **outward.**

Ans.: 3

$$\int_0^1\int_0^1\int_0^1 \mathrm{div}(F)\, dz\, dy\, dx$$

$$\mathrm{div}(F) = 2x + 2y + 2z$$

$$\int_0^1\int_0^1\int_0^1 2x + 2y + 2z \, dz\, dy\, dx$$

$$\int_0^1\int_0^1 2xz + 2yz + z^2 \Big|_0^1 dy\, dz$$

$$\int_0^1\int_0^1 2x + 2y + 1 \; dy\, dz$$

$$\int_0^1 2xy + y^2 + y \Big|_0^1 dz$$

$$\int_0^1 2x + 1 + 1 \; dx \;=\; x^2 + x + x = 1 + 1 + 1 = 3$$

11. (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) = \langle 2e^{2x+3y+4z}, 3e^{2x+3y+4z}, 4e^{2x+3y+4z} \rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \leq t \leq 1 \quad .$$

Ans: $e^{12} - 1$

$\int \frac{d}{dx} = \int 2e^{2x+3y+4z}$ $\qquad \frac{d}{dy} = 3e^{2x+3y+4z}$ $\qquad \frac{d}{dz} = 4e^{2x+3y+4z}$

$f(x,y,z) = e^{2x+3y+4z} + g(y,z)$ $\qquad f(x,y,z) = e^{2x+3y+4z} + h(z)$

$\frac{df}{dy} = 3e^{2x+3y+4z} + g_y(y,z)$ $\qquad \frac{df}{dz} = 4e^{2x+3y+4z} + h_z(z)$

$g_y(y,z) = 0$ $\qquad h_z(z) = 0$

$f(x,y,z) = e^{2x+3y+4z}$

$f(t) = e^{2t+6t+4t^2} = e^{8t+4t^2}$

$\int F \cdot dr = f(1) - f(0)$

$= e^{8+4} - e^{0}$

$= e^{12} - 1$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\ y = t\,,\ z = t^2\,,\ 0 \le t \le 1.$

Ans.: $\frac{29}{3}$

$dx = 2t \qquad dy = 1 \qquad dz = 2t$

$$\int_0^1 5t \cdot 2t + 5 \cdot t^2 \cdot 1 + 6(t^2) \cdot 2t \; dt$$

$$\int_0^1 10t + 5t^2 + 12t^3 \; dt$$

$$\frac{10t^2}{2} + \frac{5t^3}{3} + \frac{12t^4}{4} \Big|_0^1$$

$$\frac{\cancel{5}^{15}}{3} + \frac{5}{3} + \frac{\cancel{3}^{9}}{3} = \frac{29}{3}$$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z)\,|\,x^2+y^2+z^2 \le 100, z<0\} \quad .$$

Ans.: 100π

$0 \le \rho \le 10 \qquad \frac{\pi}{2} \le \phi \le \pi \qquad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{10} \frac{1}{\cancel{\rho}}\, \rho^{\cancel{2}} \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\int_0^{2\pi}\int_{\pi/2}^{\pi} \frac{\rho^2}{2}\Big|_0^{10} \sin\phi \, d\phi\, d\theta$$

$$50\int_0^{2\pi}\int_{\pi/2}^{\pi} \sin\phi \, d\phi\, d\theta$$

$$50\int_0^{2\pi} \left(-\cos\phi \Big|_{\pi/2}^{\pi}\right) d\theta$$

$$50\int_0^{2\pi} 1 \, d\theta$$

$$50(2\pi - 0) = 100\pi$$

14. (12 points) Evaluate the quadruple integral

$$\iiiint_E 360\,x\,dV \quad ,$$

where

$$E = \{(x, y, z, w) \mid 0 \le w \le 1\,,\, 0 \le z \le w\,,\, 0 \le y \le z\,,\, 0 \le x \le y\} \quad .$$

Ans.: 3

$$\int_0^1\int_0^w\int_0^z\int_0^y 360x \; dx\, dy\, dz\, dw$$

$$\int_0^1\int_0^w\int_0^z \frac{360x^2}{2}\Big|_0^y dy\, dz\, dw$$

$$\int_0^1\int_0^w\int_0^z 180y^2 \; dy\, dz\, dw$$

$$\int_0^1\int_0^w \frac{180y^3}{3}\Big|_0^z dz\, dw$$

$$\int_0^1\int_0^w 60z^3 \; dz\, dw$$

$$\int_0^1 \frac{60z^4}{4}\Big|_0^w dw = \int_0^1 15w^4\, dw = \frac{15w^5}{5}\Big|_0^1 = 3$$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u+v) \quad , \quad y = u+v+\cos(u+v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3

$\frac{dx}{du} = 6\cos(2u+v)$ $\quad$ $\frac{dy}{du} = 1-\sin(u+v)$

$\frac{dx}{dv} = 3\cos(2u+v)$ $\quad$ $\frac{dy}{dv} = 1-\sin(u+v)$

$$\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 6\cos(2u+v) & 3\cos(2u+v) \\ 1-\sin(u+v) & 1-\sin(u+v) \end{vmatrix}$$

$$= [6\cos(2u+v))(1-\sin u+v)] - [(3\cos(2u+v))(1-\sin(u+v))]$$

$$= [6\cos 0 (1-\sin 0)] - [(3\cos 0)(1-\sin 0)]$$

$$= 6 - 3 = 3$$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s): no max

Local minimum points(s): (6, 18)

saddle point(s): (0,0), $(\frac{12}{42}, \frac{72}{84})$

$f_x = 3x^2 - 6y$

$0 = 3x^2 - 6y$

$6y = 3x^2$

$y = \frac{1}{2}x^2$

$y = \frac{1}{2}(6^2)$

$y = 18$

$f_y = 2y - 6x$

$0 = x^2 - 6x$

$6x = x^2$

$x = 6$

$3x^2 - 6y = 2y - 6x$

$3x^2 + 6x = 8y$

$x = 8y \quad 3x + 6 = 8y$

$0 = 2y - 6(8y) \quad 0 = 0 - 6x$

$0 = -46y \quad x = 0$

$0 = y$

$0 = 2(\frac{3x+6}{8}) - 6x$

$0 = \frac{3}{4}x + \frac{3}{2} - \frac{\cancel{6}24}{4}x$

$\frac{21}{4}x = \frac{3}{2}$

$x = \frac{12}{42}$

$0 = 2y - 6(\frac{12}{42})$

$\frac{72}{42} = 2y$

$\frac{72}{84} = y$

potential points

(6, 18)

(0, 0)

$(\frac{12}{42}, \frac{72}{84})$

$f_{xx} = 6x \quad f_{yy} = 2 \quad f_{xy} = -6$

$D(6, 18) = 6 \cdot 6 \cdot 2 - (-6^2) = 36 \quad f_{xx}(6, 18) = 36$ local min

$D(0, 0) = 6 \cdot 0 \cdot 2 - (-6^2) = -36$ saddle

$D(\frac{12}{42}, \frac{72}{84}) = 6 \cdot \frac{12}{42} \cdot 2 - (-6^2) = \frac{144}{42} - 36$ saddle

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle x + y \; y + z \, , \, x + z \rangle \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z) \,|\, (x-1)^2 + (y+2)^2 + (z-4)^2 = 100\} \quad .$$

Div = 3

$$\iiint_R 3 \, dV$$

3 x volume of region

$$\frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$$

$$3 \times \frac{4000}{3}\pi = 4000\pi$$