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MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)

Do not write below this line

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1. (out of 12)
2. (out of 12)
3. (out of 12)
4. (out of 12)
5. (out of 12)
6. (out of 12)
7. (out of 12)
8. (out of 12)
9. (out of 12)
10. (out of 12)
11. (out of 12)
12. (out of 12)
13. (out of 12)
14. (out of 12)
15. (out of 12)
16. (out of 12)
17. (out of 8)

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tot. (out of 200)

Important note: Unlike Exams 1 and 2, you are not required to state the type of the answer, and there is no credit for stating the type. But if the given answer is the **wrong type**, you would get 0 points.

Example: Find $f'(2)$ if $f(x) = x^3$. If you give the answer $3x^2$ instead of 12, you would get **zero** points!

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) \, dA \quad .$$

1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: 0

$\rightarrow x = 10\cos t, \quad y = 10\sin t$

$\rightarrow dx = -10\sin t\,dt, \quad dy = 10\cos t\,dt$

$\rightarrow r(t) = \langle 10\cos t, 10\sin t \rangle$

$\rightarrow r'(t) = \langle -10\sin t, 10\cos t \rangle$

$\rightarrow |r'(t)| = 10$

$\rightarrow 10\int_0^{2\pi} [7(10\sin t) + 3(10\cos t)]\,dt = \boxed{0}$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $z = 5x + 5y - 5$

$\rightarrow g(x, y, z) = x^2 + 3xy + y^2 - z$

$\rightarrow g_x = 2x + 3y$

$\rightarrow g_y = 3x + 2y$

$\rightarrow g_z = -1$

$\rightarrow 5(x-1) + 5(y-1) - 1(z-5)$

$\rightarrow \boxed{z = 5x + 5y - 5}$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2\, y$ in the region

$$\{(x, y)\,|\, 0 \leq x \leq 1\,,\; 0 \leq y \leq 1 - x\,\}.$$

Absolute minimum value: 0

Absolute maximum value: 0

$\rightarrow f(0,1) = 0$

$\rightarrow$ Need to brush up on this question type

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(\,x^2\; + \; y \; + \; z\,) \quad .$$

Ans.: $f_{xxyz}(0,0,0) = -2$

$\rightarrow f(x,y,z) = \sin(x^2+y+z)$

$\rightarrow f_x = 2x\cos(x^2+y+z)$

$\rightarrow f_{xx} = 2\cos(x^2+y+z) - 4x^2\sin(x^2+y+z)$

$\rightarrow f_{xxy} = -2\sin(x^2+y+z) - 4x^2\cos(x^2+y+z)$

$\rightarrow f_{xxyz} = 4x^2\sin(x^2+y+z) - 2\cos(x^2+y+z)$

$\rightarrow \boxed{f_{xxyz}(0,0,0) = -2}$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1, 1, 1)$ if (x, y, z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: $\frac{dz}{dy} = -1$

$\rightarrow F(x,y,z) = xy + xz + yz + x^2y^2z^2 - 4$

$\rightarrow dz/dy = -F_y/F_z$

$\rightarrow F_y = x + z + 2x^2yz^2$

$\rightarrow F_z = x + y + 2x^2y^2z$

$\rightarrow F_y(1,1,1) = 4$

$\rightarrow F_z(1,1,1) = 4$

$\rightarrow \boxed{dz/dy = -1}$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t\,,\; y = 2 + t\,,\; z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

and the line

$$x = -t\,,\; y = 1 + t\,,\; z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: $z = y - 1$

$\to n = \langle 1,1,1 \rangle$, $n_2 = \langle -1,1,1 \rangle$

$\to n_1 \times n_2 = \langle 0,-2,2 \rangle$ (Maple)

$\to -2(y-2) + 2(z-3) = 0$

$\to \boxed{z = y - 1}$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle\, -4\,\sin 2t\,,\; -4\,\cos 2t\,,\; 9e^{3t}\,\rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3\rangle$ and its position at $t = 0$ is $\langle 0, 1, 1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $x\left(\frac{\pi}{4}\right) = \langle \frac{\pi}{2}+1,\ 1,\ e^{3\pi/4} + \frac{3\pi}{4} + 1 \rangle$

$\rightarrow v(t) = \int a(t)\,dt = \int \langle -4\sin(2t), -4\cos(2t), 9e^{3t} \rangle\,dt$

$\rightarrow \int \langle -4\sin(2t), -4\cos(2t), 9e^{3t} \rangle\,dt = \langle 2\cos(2t), -2\sin(2t), 3e^{3t} \rangle + \langle 2, 0, 3 \rangle$

$\rightarrow v(t) = \langle 2\cos(2t) + 2,\ -2\sin(2t),\ 3e^{3t} + 3 \rangle$

$\rightarrow x(t) = \int v(t)\,dt = \int \langle 2\cos(2t) + 2,\ -2\sin(2t),\ 3e^{3t} + 3 \rangle\,dt$

$\rightarrow \int \langle 2\cos(2t) + 2,\ -2\sin(2t),\ 3e^{3t} + 3 \rangle\,dt = \langle \sin(2t) + 2t,\ \cos(2t) + 1,\ e^{3t} + 3t + 1 \rangle$

$\rightarrow \boxed{x\left(\frac{\pi}{4}\right) = \langle \frac{\pi}{2}+1,\ 1,\ e^{3\pi/4} + \frac{3\pi}{4} + 1 \rangle}$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2z)\, ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: $\frac{7\sqrt{6}}{2}$

$\rightarrow f(x,y,z) = x+y+2z$

$\rightarrow f(r(t)) = 7t$

$\rightarrow r'(t) = \langle 1,1,2 \rangle, \quad |r'(t)| = \sqrt{6}$

$\rightarrow \int_0^1 7\sqrt{6}\, t\, dt = \boxed{\frac{7\sqrt{6}}{2}}$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3}f(x,y,z))\cos(\frac{\pi}{4}g(x,y,z))$$

Ans.: 0

$$\rightarrow \lim_{(x,y,z)\to(1,1,1)} \left(\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)\right) = \frac{\sqrt{3}}{2}\cdot 0 = \boxed{0}$$

10. (12 points) Compute

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\, y^2 + xz^3\,,\, z^2 + e^{xy}\,\rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \le x,y,z \le 1\}$$

with the normal pointing **outward**.

Ans.: 3

$\rightarrow \mathrm{div}(F) = 2x + 2y + 2z$

$\rightarrow \int_0^1\int_0^1\int_0^1 \mathrm{div}(F)\,dz\,dy\,dx = \boxed{3}$

11. (12 points) By finding a function f such that $\mathbf{F} = \bigtriangledown f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x,y,z) = \langle\, 2e^{2x+3y+4z} \,,\, 3e^{2x+3y+4z} \,,\, 4e^{2x+3y+4z} \,\rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: $e^{12} - 1$

$\rightarrow \int 2e^{2x+3y+4z}\,dx = e^{2x+3y+4z} + g(y,z)$

$\rightarrow$ Did other work on external scrap paper

$\rightarrow r(t) = \langle t, 2t, t^2 \rangle$

$\rightarrow r(0) = \langle 0,0,0 \rangle \,,\; r(1) = \langle 1,2,1 \rangle$

$\rightarrow f(1,2,1) - f(0,0,0) = \boxed{e^{12} - 1}$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\ y = t\,,\ z = t^2\,,\ 0 \le t \le 1.$

Ans.: 8

$$\rightarrow \int_0^1 (10t^2 + 5t^2 + 12t^3)\,dt = \int_0^1 (15t^2 + 12t^3)\,dt = \boxed{8}$$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z) \,|\, x^2+y^2+z^2 \le 100, z < 0\} \quad .$$

Ans.: 100π

$\to \frac{\pi}{2} \le \phi \le \pi \, ; \quad 0 \le \theta \le 2\pi \, ; \quad 0 \le \rho \le 10$

$\to \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{10} \rho \sin\phi \, d\rho \, d\phi \, d\theta = \boxed{100\pi}$

14. (12 points) Evaluate the quadruple integral

$$\int\int\int\int_E 360\,x\,dV \quad ,$$

where

$$E = \{(x,y,z,w)\,|\,0 \le w \le 1\,,\,0 \le z \le w\,,\,0 \le y \le z\,,\,0 \le x \le y\} \quad .$$

Ans.: 15

$\rightarrow \int_0^1\int_0^w\int_0^z\int_0^y (360x)\,dx\,dy\,dz\,dw$

$\rightarrow \int_0^y 360x\,dx = 180y^2$

$\rightarrow \int_0^z 180y^2\,dy = 60z^3$

$\rightarrow \int_0^w 60z^3\,dz = 15w^4$

$\rightarrow \int_0^1 15w^4\,dw = \boxed{15}$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3

$\rightarrow x_u = 6\cos(2u+v)$

$\rightarrow x_v = 3\cos(2u+v)$

$\rightarrow y_u = 1 - \sin(u+v)$

$\rightarrow y_v = 1 - \sin(u+v)$

$\rightarrow x_u = 6, \quad x_v = 3, \quad y_u = 1, \quad y_v = 1$

$\rightarrow J = 6 - 3 = \boxed{3}$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s): none

Local minimum points(s): $(6,18)$

saddle point(s): $(0,0)$

$\to f(x,y) = x^3 + y^2 - 6xy$

$\to f_x = 3x^2 - 6y$

$\to f_y = 2y - 6x$

$\to f_{xx} = x, \quad f_{yy} = 4, \quad f_{xy} = -6$

$\to$ Critical Points @ $(0,0)$ and $(6,18)$

$\to -36$; $D < 0$ for $(0,0)$ so saddle point

$\to 108$; $D > 0$ for $(6,18)$ so local minimum

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\,\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle\, x + y\ y + z\,,\ x + z\,\rangle \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z) \,|\, (x-1)^2\ +\ (y+2)^2\ +\ (z-4)^2 = 100\} \quad .$$

$\rightarrow \mathrm{div}(F) = 3$

$\rightarrow \iiint 3\, dV$

$\rightarrow \frac{4}{3}\pi R^3 \Rightarrow \boxed{V = 4000\pi}$