1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: 400π

$x(t) = 10\cos\theta$

$y(t) = 10\sin\theta$

$x'(t) = -10\sin\theta$

$y'(t) = 10\cos\theta$

$$\int_C 7y\,dx + 3x\,dy$$

$$= \int_0^{2\pi} (7)(10\sin\theta)(-10\sin\theta) + (3)(10\cos\theta)(10\cos\theta)\,d\theta$$

$$= 100\int_0^{2\pi} 3\cos^2\theta - 7\sin^2\theta$$

$$= 300\int_0^{2\pi} \cos^2\theta \quad - \quad 700\int_0^{2\pi} \sin^2\theta$$

2nd Integral

$$-700\int_0^{2\pi} \frac{1}{2} - \frac{\cos 2\theta}{2}$$

$$= -700\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -700\pi$$

1st Integral

$\cos 2\theta = 2\cos^2\theta - 1$

$\dfrac{\cos 2\theta + 1}{2} = \cos^2\theta$

$$300\int_0^{2\pi} \frac{\cos 2\theta}{2} + \frac{1}{2}$$

$$= 300\left[\frac{\sin 2\theta}{4} + \frac{1}{2}\theta\right]$$

$$= 300\pi$$

$300\pi - 700\pi = -400\pi$

$-400\pi\,(-1) = \boxed{400\pi}$

2. **(12 points) Find an equation of the tangent plane to the surface**

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $z = 5x + 5y - 5$

$\cancel{f}\ f(x,y) = x^2 + 3xy + y^2 \quad f(1,1) = 5$

$f_x = 2x + 3y \quad f_x(1,1) = 5$

$f_y = 3x + 2y \quad f_y(1,1) = 5$

$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$z = 5 + 5(x-1) + 5(y-1)$

$z = 5 + 5x - 5 + 5y - 5$

$\boxed{z = 5x + 5y - 5}$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x,y) = x^2 y$ in the region

$$\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}.$$

Absolute minimum value: 0

Absolute maximum value: $\frac{4}{27}$

$f_x = 2xy = 0$

$f_y = x^2 = 0$

$x, y = 0$

$f(0,0) = 0$

Left

$x = 0$

$0 \le y \le 1-x$

$f(0,y) = 0$

$F' = 0$

$F(0) = 0$

$-x) = 0$

Right

$x = 1$

$0 \le y \le 1-x$

$f(1,y) = y = F$

$F' = 1 \to 0$

$F(0) = 0$

$F(1-x) = 1-x$

$x = 1 \to = 0$

Down

$y = 0$

$0 \le x \le 1$

$f(x,0) = 0$

Top:

$y = 1-x$

$0 \le x \le 1$

$f(x, 1-x) = x^2 - x^3 = F(x)$

$F'(x) = 2x - 3x^2, x = 0, 2/3$

$F(0) = 0$

$F(2/3) = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}$

$F(0) = 0$

$F(1) = 0$

Min: 0

Max: $\frac{4}{27}$

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(x^2 + y + z) \quad .$$

Ans.: 0

$f_x = 2x\cos(x^2+y+z)$

$f_{xx} = -(2x)2x\sin(x^2+y+z)$

$= -4x^2\sin(x^2+y+z)$

$f_{xxy} = -4x^2\cos(x^2+y+z)$

$f_{xxyz} = 4x^2\sin(x^2+y+z)$

$4x^2\sin(x^2+y+z)\Big|_{x=0,y=0,z=0} = \boxed{0}$

5. **(12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1,1,1)$ if (x,y,z) are related by:**

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: -1

$$xy + xz + yz + x^2y^2z^2 = 4$$

$$x + \frac{\partial z}{\partial y} + \left(y\frac{\partial z}{\partial y} + z\right) + x^2\left(y^2 2z\frac{\partial z}{\partial y} + 2z^2 y\right) = 0$$

$$\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} + x^2y^2 2z\frac{\partial z}{\partial y} = -x - z - 2x^2yz^2$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-x-z-2x^2yz^2}{1+y+x^2y^2 2z}} \longrightarrow \frac{-1-1-(2)(1)(1)(1)}{1+1+(1)(1)(2)(1)}$$

$$= \frac{-4}{4} = \boxed{-1}$$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t,\ y = 2 + t,\ z = 3 + t \quad (-\infty < t < \infty)\ ,$$

and the line

$$x = -t,\ y = 1 + t,\ z = 2 + t \quad (-\infty < t < \infty)\ .$$

Ans.: $-y + z = 1$

$r_1(t) = \langle 1, 2, 3 \rangle + t\langle 1, 1, 1 \rangle$

$r_2(t) = \langle 0, 1, 2 \rangle + t\langle -1, 1, 1 \rangle$

$\langle 1, 1, 1 \rangle \times \langle -1, 1, 1 \rangle = \langle 0, -2, 2 \rangle$

$r(0) = (1, 2, 3)$

$0(x-1) - 2(y-2) + 2(z-3)$

$-2y + 4 + 2z - 6 = 0$

$\boxed{-y + z = 1}$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4\sin 2t\,,\, -4\cos 2t\,,\, 9e^{3t}\rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3\rangle$ and its position at $t = 0$ is $\langle 0, 1, 1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $\langle 1 + \pi/4, 0, e^{3\pi/4} \rangle$

$v(t) = \int a(t) = \langle 2\cos 2t, -2\sin 2t, 3e^{3t} \rangle + C$

$\langle 1, 0, 3 \rangle + C = \langle 2, 0, 3 \rangle$

$C = \langle 1, 0, 0 \rangle$

$v(t) = \langle 2\cos 2t + 1, -2\sin 2t, 3e^{3t} \rangle$

$s(t) = \int v(t) = \langle \sin 2t + t, \cos 2t, e^{3t} \rangle + C$

$\langle 0, 1, 1 \rangle + C = \langle 0, 1, 1 \rangle \qquad C = 0$

$s(t) = \langle \sin 2t + t, \cos 2t, e^{3t} \rangle$

$\boxed{s(\pi/4) = \langle 1 + \pi/4, 0, e^{3\pi/4} \rangle}$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2\,z)\,ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \ .$$

Ans.: $\frac{21}{2}$

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add z-component

$$\int_C f(x,y)\,ds = \int f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}$$

$x'(t) = 1$

$y'(t) = 2$

$z'(t) = 2$

$$= \int_0^1 t + 2t + 4t \sqrt{9} \; dt$$

$$= 21 \int_0^1 t$$

$$= 21 \left. \frac{t^2}{2} \right|_0^1 = \boxed{\frac{21}{2}}$$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3} f(x,y,z)) \cos(\frac{\pi}{4} g(x,y,z))$$

Ans.: 0

$$\lim_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3} f(x,y,z)\right) \cos\left(\frac{\pi}{4} (g(x,y,z)\right)$$

$$= \sin\left(\frac{\pi}{3}\left(\lim_{(x,y,z)\to(1,1,1)} f(x,y,z)\right)\right) \cos\left(\frac{\pi}{4}\left(\lim_{(x,y,z)\to(1,1,1)} g(x,y,z)\right)\right)$$

$$= \left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{2}\right)$$

$$\left(\frac{\sqrt{3}}{2}\right)(0) = \boxed{0}$$

10. (12 points) Compute

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\, y^2 + xz^3\,,\, z^2 + e^{xy}\,\rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \le x,y,z \le 1\}$$

with the normal pointing **outward**.

Ans.: 3

$$\iint_S F \cdot ds = \iiint div(F)\, dV$$

$$div(F) = 2x + 2y + 2z$$

$$= 2\int_0^1\int_0^1\int_0^1 x+y+z$$

$$(2) = \left[\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right]\Big|_0^1$$

$$= (2)\left(\frac{3}{2}\right) = \boxed{3}$$

11. (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) = \langle 2e^{2x+3y+4z}, 3e^{2x+3y+4z}, 4e^{2x+3y+4z} \rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: $e^{12} - 1$

@ $t=1$: $(1,2,1)$
@ $t=0$: $(0,0,0)$

on inspection, $f = e^{2x+3y+4z}$

By fundamental theorem of line integrals: $f(1,2,1) - f(0,0,0)$

$$\boxed{= e^{12} - 1}$$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C: x = t^2\,,\, y = t\,,\, z = t^2\,,\, 0 \le t \le 1.$

Ans.: $47/6$

$x'(t) = 2t$

$y'(t) = 1$

$z'(t) = 2t$

$x(t) = t^2$

$y(t) = t$

$z(t) = t^2$

$$\int_0^1 5t(2t) + 5t + 6t^2(2t)$$

$$= \int_0^1 12t^3 + 10t^2 + 5t$$

$$= 3t^4 + \frac{10}{3}t^3 + \frac{5}{2}t^2 \Big|_0^1$$

$$= 3 + \frac{10}{3} + \frac{5}{2}$$

$$= 3 + \frac{25}{6}$$

$$\boxed{= \frac{47}{6}}$$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z)\,|\,x^2+y^2+z^2 \le 100, z<0\} \quad .$$

Ans.: 100π

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \iiint \frac{1}{\rho}\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$0 \le \rho \le 10$

$0 \le \theta \le 2\pi$

$\frac{\pi}{2} \le \varphi \le \pi$

$$= \int_0^{2\pi} d\theta \int_0^{10} \rho\, d\rho \int_{\frac{\pi}{2}}^{\pi} \sin\varphi\, d\varphi$$

$$= (2\pi)\left(\frac{\rho^2}{2}\Big|_0^{10}\right)\left(-\cos\varphi\Big|_{\pi/2}^{\pi}\right) \rightarrow -\cos\pi + \cos\frac{\pi}{2} = 1+0$$

$$= (100\pi)(1+0)$$

$$\boxed{= 100\pi}$$

14. (12 points) Evaluate the quadruple integral

$$\iiiint_E 360\, x \, dV \quad ,$$

where

$$E = \{(x, y, z, w) \,|\, 0 \overset{4}{\le} w \le 1 \,,\, 0 \overset{3}{\le} z \le w \,,\, 0 \overset{2}{\le} y \le z \,,\, 0 \overset{1}{\le} x \le y\} \quad .$$

Ans.: 3

$$\int_0^1 \int_0^w \int_0^z \int_0^y 360x \, dx\, dy\, dz\, dw$$

$$= \int_0^1 \int_0^w \int_0^z 180x^2 \Big|_0^y \, dy\, dz\, dw$$

$$= \int_0^1 \int_0^w \int_0^z 180y^2 \, dy\, dz\, dw$$

$$= \int_0^1 \int_0^w 60z^3 \, dz\, dw$$

$$= \int_0^1 15w^4 \, dw$$

$$= 3w^5 \Big|_0^1 = \boxed{3}$$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix} = 6 - 3 = \boxed{3}$$

$x_u = 6\cos(2u+v) = 6$

$x_v = 3\cos(2u+v) = 3$

$y_u = 1 - \sin(u+v) = 1$

$y_v = 1 - \sin(u+v) \quad 1$

16. (12 points) **Find the local maximum and minimum points and saddle point(s) of the function** $f(x,y) = x^3 + y^2 - 6xy$

Local maximum points(s): None

Local minimum points(s): (6,18)

saddle point(s): (0,0)

$f_x = 3x^2 - 6y$

$f_y = 2y - 6x$

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = -6$

$3x^2 - 6y = 0$

$2y - 6x = 0$

$y = 3x$

$3x^2 - 18x = 0$

$3x(x-6) = 0$

$x = 0, 6$

$y = 0, 18$

Critical points: (0,0)(6,18)

(0,0)

$f_{xx} = 0$

$D = f_{xx} f_{yy} (f_{xy})^2$

$= 0 - (-6)^2$

$= -36$

(0,0) is a saddle point

(6,18)

$f_{xx} = 36$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (36)(2) - (-6)^2$

$= 72 - 36$

$= 36$

(6,18) is a local minimum

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle x + y, y + z, x + z \rangle \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z) \,|\, (x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 100\} \quad .$$

Answer: 4000π

$$\iint_S F \cdot ds = \iiint_V div(F)\, dv$$

$$div(F) = 1 + 1 + 1 = 3 \qquad \iiint_V 3\, dv = (\text{Integrand})(\text{volume})$$

$$= \left(\frac{4000}{3}\pi\right)(3)$$

$$\text{volume} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1000) \qquad \boxed{= 4000\pi}$$