Name: SHUBIN XIE

section: 22 E-mail address: sx111 @rutgers.edu.

Math 251 (22) Dr. Z, practice for final exam. Dec, 13, 2020,

dorm room, 14:40 ~~PM~~ 18:10 PM (china time).

Ans:

1, $\cancel{1}400\pi$.

2, $5x+5y-z=5$. $z=5x+5y-5$.

3, max: $\frac{4}{27}$ min: 0.

4, -2.

5, -1.

6, $-y+z=1$

7, $(1, 0, e^{\frac{3}{4}\pi})$.

8, $\frac{21}{2}$.

9, 0.

10, 3.

11, $e^{12}-1$

12, 8.

13, 100π.

14, 3.

15, 3.

16, max point: none min point: (6, 18) saddle point (0, 0).

17, 4000π.

total: 200/200 I have more 100 min after I finish.

finish time: 16:30 PM.

I can check. (enough time to check).

1, Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy.$$

where C is the circle $x^2+y^2=100$ traveld in the clockwise direction.

Ans: Use green theorem

$$-\iint \frac{d}{dx}3x - \frac{d}{dy}7y \; dA = -\iint 3-7\,dA = -\iint -4\,dA = +4\iint dA$$

Area: $\pi r^2 = 100\pi$

$$\int_C 7y\,dx+3x\,dy = +400\pi$$

Ans: $+400\pi$

2. Find an equation of the tangent plane to the surface.

$$z = x^2+3xy+y^2.$$

at the point $(1,1,5)$.

Ans: $5 = 1+3+1 = 5$.

point $(1,1,5)$ on the surface.

$x^2+3xy+y^2-z=0$

normal vector $= \langle 2x+3y,\ 3x+2y,\ -1\rangle = \langle 2+3,\ 3+2,\ -1\rangle$

$= \langle 5,\ 5,\ -1\rangle$

$\langle 5,5,-1\rangle \cdot \langle x-1,\ y-1,\ z-5\rangle = 0$

$5(x-1) + 5(y-1) - (z-5) = 0$

$5x-5 + 5y-5 - z+5 = 0$

$5x+5y-z = 5$.

Ans: $5x+5y-z=5$.

$z = 5x+5y-5$.

3、Find the absolute maximum value and the absolute minimum value of the function $f(x,y) = x^2 y$ in the region.

$$\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

Ans: $f_x = 2xy$, $f_y = x^2$

$2xy = 0$ $x^2 = 0$

$x = 0$, $y = 0$ or 1

absolute maximum value:

$\frac{4}{27}$

absolute minimum value:

0

$f(0,0) = 0$

$f(0,1) = 0$.

① $x = 0,\ 0 \le y \le 1-x$.

$\cancel{f(x,y) = 0}$.

$f(0,y) = 0$.

max: 0

min: 0.

② $0 \le x \le 1$, $y = 0$.

$f(x,0) = 0$.

max: 0

min: 0

③ $x = 1$ $0 \le y \le 1-x$

$f(1,y) = y$

$f'(1,y) = 1$

no critical point

$f(1,0) = 0$

$f(1,1-x) = 1^2 \cdot 0 = 0$.

max: 0

min: 0.

④ $y = 1-x$ $0 \le x \le 1$

$f(x,1-x) = x^2 \cdot (1-x)$
$= x^2 - x^3$

$f'(x,1-x) = 2x - 3x^2 = 0$

$x(2-3x) = 0$

$x_1 = 0$ $x_2 = \frac{2}{3}$

$F(0) = 0$ $F(\frac{2}{3}) = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}$

$F(0) = 0$ $F(1) = 1 - 1 = 0$.

max: $\frac{4}{27}$ min: 0.

4. compute $f_{xxyz}(0,0,0)$ (in other words $\frac{d^4}{dx^2dydz} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z)=\sin(x^2+y+z).$$

$f_x = \cos(x^2+y+z)\cdot 2x$

$f_{xx} = -\sin(x^2+y+z)\cdot 2x\cdot 2x + \cos(x^2+y+z)\cdot 2.$

$\quad = -4x^2\cdot\sin(x^2+y+z) + 2\cos(x^2+y+z).$

$f_{xxy} = -4x^2\cdot\cos(x^2+y+z)\cdot 1 + 2\cdot -\sin(x^2+y+z)\cdot 1$

$\quad = -4x^2\cdot\cos(x^2+y+z) - 2\sin(x^2+y+z)$

$f_{xxyz} = -4x^2\cdot -\sin(x^2+y+z)\cdot 1 - 2\cos(x^2+y+z)\cdot 1.$

$\quad = 4x^2\cdot\sin(x^2+y+z) - 2\cos(x^2+y+z).$

plug in $(0,0,0)$

$f_{xxyz} = 0 - 2\cos(0+0+0)$

$\quad = -2\cos 0$

$\quad = -2\cdot 1$

$\quad = -2.$

Ans: -2

5. Find $\frac{dz}{dy}$ at the point $(1,1,1)$ if (x,y,z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4.$$

Ans:

$\frac{dz}{dy}$:

$$x + x\cdot z' + z + yz' + 2x^2yz^2 + 2x^2y^2 z\cdot z' = 0.$$

$$(x + y + 2x^2y^2z)z' = -x - z - 2x^2yz^2.$$

$$z' = \frac{-x - z - 2x^2yz^2}{x + y + 2x^2y^2z}$$

plug $(1,1,1)$

$$z' = \frac{-1-1-(2\cdot 1^2\cdot 1\cdot 1^2)}{1+1+(2\cdot 1^2\cdot 1^2\cdot 1)}$$

$$= \frac{-2-2}{2+2}$$

$$= \frac{-4}{4}$$

$$= -1.$$

$z' = -1$

$\frac{dz}{dy} = -1$

Ans: -1.

6, Find an equation for the plane that contains both the line

$$x = 1+t, \; y = 2+t, \; z = 3+t \; (-\infty < t < \infty).$$

and the line

$$x = -t, \; y = 1+t, \; z = 2+t \; (-\infty < t < \infty).$$

Ans:

L_1: $\langle 1, 2, 3\rangle + t\langle 1, 1, 1\rangle$

L_2: $\langle 0, 1, 2\rangle + t\langle -1, 1, 1\rangle$

point: $t=0$ $\cancel{\langle 1,2,3\rangle}$ $(1,2,3)$.

$\langle 1,1,1\rangle \times \langle -1,1,1\rangle = i(1-1) - j(1-(-1)) + k(1-(-1))$

$$\begin{matrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{matrix} \qquad = \langle 0, -2, 2\rangle$$

$\langle 0,-2,2\rangle \cdot \langle x-1, y-2, z-3\rangle = 0$.

$-2(y-2) + 2(z-3) = 0$

$\Rightarrow -y+2+z-3=0$

$\Rightarrow -y+z-1=0$

$-y+z=1.$

Ans: $-y+z=1.$

7. A certain particle has acceleration given by

$$a(t) = \langle -4\sin 2t, -4\cos 2t, 9e^{3t} \rangle.$$

If its velocity at $t=0$ is $\langle 2,0,3\rangle$ and its position at $t=0$ is $\langle 0,1,1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans: $v(t) = \int a(t) = \langle \cancel{4\cos t}\ 2\cos 2t, -2\sin 2t, 3e^{3t} \rangle$

$\cancel{t=0}$

$v(0) = \langle 2, 0, 3 \rangle$

$C = 0.$

$r(t) = \int v(t) = \langle \sin 2t, \cos 2t, e^{3t} \rangle$

$r(0) = \langle 0, 1, 1 \rangle$

$C = 0.$

$r(\frac{\pi}{4}) = \langle \sin \frac{\pi}{2}, \cos \frac{\pi}{2}, e^{\frac{3}{4}\pi} \rangle$

$= \langle 1, 0, e^{\frac{3}{4}\pi} \rangle.$

Ans: $(1, 0, e^{\frac{3}{4}\pi})$.

8. Compute the (scalar-function) line-integral.

$$\int_C (x+y+z)\,ds$$

where the curve C is given by ~~the~~ the parametric equation.

$$r(t) = \langle t, 2t, 2t \rangle \quad 0 \le t \le 1.$$

Ans:

$$r'(t) = \langle 1, 2, 2 \rangle$$

$$ds = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$$

$$\int_0^1 (t+2t+2t\cdot 2)\,3\,dt$$

$$= \int_0^1 (3t+4t)\,3\,dt$$

$$= \int_0^1 7t\cdot 3\,dt$$

$$= \int_0^1 21t\,dt$$

$$= \frac{21}{2}t^2 \Big|_0^1$$

$$= \frac{21}{2}\cdot 1$$

$$= \frac{21}{2}$$

Ans: $\frac{21}{2}$

9. If $\lim\limits_{(x,y,z)\to(1,1,1)} f(x,y,z)=1$, $\lim\limits_{(x,y,z)\to(1,1,1)} g(x,y,z)=2$.

compute $\lim\limits_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3} f(x,y,z)\right)\cos\left(\frac{\pi}{4} g(x,y,z)\right)$.

Ans: $\lim\limits_{(x,y,z)\to(1,1,1)} \sin\left(\frac{\pi}{3}\cdot 1\right)\cos\left(\frac{\pi}{\cancel{4}_{2}}\cdot \cancel{2}\right)$

$= \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{2}$

$= 0$.

Ans: 0

?10. Compute

$$\iint_S F \cdot ds$$

where

$$F = \langle x^2 + \sin(y+z),\ y^2 + xz^3,\ z^2 + e^{xy} \rangle$$

and where S is the boundary (consisting of all six faces) of the cube.

$$\{ (x,y,z) \mid 0 \le x, y, z \le 1 \}$$

with the normal pointing outward.

Ans:

$$\operatorname{div} F = 2x + 2y + 2z$$

$$\int_0^1\int_0^1\int_0^1 2x + 2y + 2z \; dz\,dy\,dx$$

$$= \int_0^1\int_0^1 2xz + 2yz + z^2 \Big|_0^1 dy\,dx$$

$$= \int_0^1\int_0^1 2x + 2y + 1 \; dy\,dx$$

$$= \int_0^1 2xy + y^2 + y \Big|_0^1 dx$$

$$= \int_0^1 2x + 1 + 1 \; dx$$

$$= x^2 + x + x \Big|_0^1$$

$$= 1 + 1 + 1$$

$$= 3.$$

Ans: 3.

11, By finding a function f such that $F=\nabla f$, evaluate $\int_C F\cdot dr$ along the given curve.

$$F(x,y,z)=\langle 2e^{2x+3y+4z},\ 3e^{2x+3y+4z},\ 4e^{2x+3y+4z}\rangle,$$

$$C:\ x=t,\ y=2t,\ z=t^2,\ 0\le t\le 1.$$

Ans:

$\cancel{f(x)}$

$\cancel{f(x,y,z)}$

$f\_x = 2e^{2x+3y+4z}$

$f = \cancel{2}e^{2x+3y+4z} + g(y,z)$

$f\_y = 3e^{2x+3y+4z} = 3e^{2x+3y+4z}$

$\Leftarrow g(y,z)=0.$

$f = \cancel{\cdot}e^{2x+3y+4z} + h(z)$

$f\_z = 4e^{2x+3y+4z} + h'(z) = 4e^{2x+3y+4z}$

$h'(z)=0$

$h(z)=0.$

$f = e^{2x+3y+4z}.$

curl (grad f) = $\langle 0,0,0\rangle$

conservative.

$F=\nabla f$

$\int_C F\cdot dr = f(\text{end point}) - f(\text{start point})$

$t=1$

$x=1,\ y=2,\ z=1.$

$t=0$

$x=0,\ y=0,\ z=0.$

plug endpoint $(1,2,1)$

start point $(0,0,0)$

$$\int_C F\,dr = e^{2+6+4} - e^0$$

$$= e^{12}-1$$

Ans: $e^{12}-1$.

12, Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz.$$

where $C: x=t^2 \quad y=t \quad z=t^2 \quad 0 \le t \le 1.$

Ans:

$dx = 2t\,dt \quad dy = dt \quad dz = 2t\,dt$

$$\int_0^1 5t \cdot 2t\,dt + 5t^2 \cdot dt + 6t^2 \cdot 2t\,dt$$

$$= \int_0^1 10t^2\,dt + 5t^2\,dt + 12t^3\,dt$$

$$= \int_0^1 15t^2 + 12t^3\,dt$$

$$= 5t^3 + 3t^4 \Big|_0^1$$

$$= 5\cdot 1 + 3 \cdot 1$$

$$= 5+3$$

$$= 8.$$

Ans: 8.

7.7.13. Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV$$

where E is the hemisphere

$$\{(x,y,z) \mid x^2+y^2+z^2 \le 100,\ z<0\}.$$

Ans:

change it to sphere coordinate:

$$\{(r,t,f) \mid r=0..10,\ t=0..2\pi,\ f=\tfrac{\pi}{2}..\pi\}.$$

$$\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\int_0^{10} \frac{1}{\sqrt{[r\sin(f)\cos(t)]^2+[r\sin(f)\sin(t)]^2+[r\cos(f)]^2}} \cdot r^2\sin(f)\,dr\,dt\,df.$$

$= 100\pi$.

Ans: 100π.

14, Evaluate the quadruple integral.

$$\iiiint_E 360x\,dV$$

where

$$E=\{(x,y,z,w)\mid 0\le w\le 1,\ 0\le z\le w,\ 0\le y\le z,\ 0\le x\le y\}$$

$$\int_0^1\int_0^w\int_0^z\int_0^y 360x\,dx\,dy\,dz\,dw$$

$$=\int_0^1\int_0^w\int_0^z 180x^2\Big|_0^y\,dy\,dz\,dw$$

$$=\int_0^1\int_0^w\int_0^z 180y^2\,dy\,dz\,dw$$

$$=\int_0^1\int_0^w 60y^3\Big|_0^z\,dz\,dw$$

$$=\int_0^1\int_0^w 60z^3\,dz\,dw$$

$$=\int_0^1 15z^4\Big|_0^w\,dw$$

$$=\int_0^1 15\cancel{z}w^4\,dw$$

$$=\cancel{15}\,3w^5\Big|_0^1$$

$$=3\cdot 1^5$$

$$=3\cdot 1$$

$$=3.$$

Ans: 3.

7.15, Find the Jacobian of the transformation from (u,v)-space to (x,y)-space

$$x=3\sin(2u+v) \qquad y=u+v+\cos(u+v),$$

at the point $(u,v)=(0,0)$

Ans:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{dx}{du}\cdot\frac{dy}{dv} - \frac{dx}{dv}\cdot\frac{dy}{du}$$

$$\frac{dx}{du} = 2\cdot 3\cos(2u+v) = 6\cos(2u+v) = 6\cos 0 = 6$$

$$\frac{dx}{dv} = 1\cdot 3\cos(2u+v) = 3\cos(2u+v) = 3\cos 0 = 3$$

$$\frac{dy}{du} = 1\cancel{x} - \sin(u+v) = 1-\sin 0 = 1$$

$$\frac{dy}{dv} = 1-\sin(u+v) = 1-\sin(0) = 1.$$

$$\text{Jacobian} = \frac{6\cdot 1 - 3\cdot 1}{\cancel{6\cos(2u}}$$

$$= 6-3$$

$$= 3.$$

Ans: 3.

16. Find the local maximum and minimum points and saddle points(s) of the function $f(x,y) = x^3 + y^2 - 6xy$.

Ans:

$$f_x = 3x^2 - 6y$$

$$f_y = 2y - 6x$$

$$f_{xx} = 6x$$

$$f_{yy} = 2$$

$$f_{xy} = -6.$$

$\cancel{3x^2-6x}$ $3x^2 - 6y = 0$ $\qquad$ $2y - 6x = 0$

$x^2 - 2y = 0$ $\qquad$ $y - 3x = 0$

$x^2 - 6x = 0$ $\qquad$ $y = 3x$

$x(x-6) = 0$

$x_1 = 0$ $\qquad$ $y_1 = 0$

$x_2 = 6$ $\qquad$ $y_2 = 18$

$(0,0)$, $(6,18)$ is critical point.

$(0,0)$ $\quad f_{xx} = 0 \quad f_{yy} = 2 \quad f_{xy} = -6$

$D = 0 \cdot 2 - (-6)^2 = -36 < 0$ $\quad$ saddle point.

$(6,18)$ $\quad f_{xx} = 36 \quad f_{yy} = 2 \quad f_{xy} = -6$.

$D = 36 \cdot 2 - 36 = 36 > 0$

$\cancel{f_{xx}}$ $f_{xx} = 36 > 0$ $\qquad$ local min point.

$\cancel{(0,0)}$

Ans:

Local max point: none

Local min point: $(6, 18)$

Saddle point: $(0,0)$.

17. Use the divergence Theorem to calculate the surface integral $\iint_S F \cdot dS$, where.

$$F(x,y,z) = \langle x+y,\ y+z,\ x+z \rangle.$$

where S is the sphere (center (1, −2, 4) and radius 10), in other words the region in 3D space:

$$\{(x,y,z) \mid (x-1)^2 + (y+2)^2 + (z-4)^2 = 100\}.$$

$$\text{div } F = 1+1+1 = 3.$$

$$\iint_S F \cdot dS = \iiint_E 3\, dV = 3 \cdot \text{volume}$$

$$= 3 \cdot \tfrac{4}{3}\pi\, 1000$$

$$= 4000\pi$$

Ans: 4000π.