NAME: (print!) Yongshan Li
Section: 23

1. Find the Jacobian of the transformation from (u,v,w)-space to (x,y,z)-space

$$x = uv + w, \quad y = uw + v, \quad z = vw + u$$

at the point $(u,v,w) = (7,1,5)$.

$x_u = v, \quad x_v = u, \quad x_w = 1$

$y_u = w, \quad y_v = 1, \quad y_w = u$

$z_u = 1, \quad z_v = w, \quad z_w = v$

$$\text{Jacobian} = \begin{bmatrix} v, & u, & 1 \\ w, & 1, & u \\ 1, & w, & v \end{bmatrix} = \begin{bmatrix} 1, & 7, & 1 \\ 5, & 1, & 7 \\ 1, & 5, & 1 \end{bmatrix}$$

$$= 1(-34) - 7(-2) + 1(24)$$

$$= 4$$

Ans. 4

2. (i) Show that

$F = \langle 9x^2yz + yz + \cos(x+y+z), 3x^3z + xz + \cos(x+y+z),$
$3x^3y + xy + \cos(x+y+z)\rangle$, is a conservative vector field

(ii) Find a function $f(x,y,z)$ such that $F = \nabla f$.

(iii) Find the line-integral $\int_C F \cdot dr$ where C is the curve

$r = \langle \sin t, \cos t + 9, \sin t\rangle,\ 0 \le t \le \pi$

Ans. (ii) $f = 3x^3yz + xyz + \cos(x+y+z)$ (iii) 0.694.

(i)

$$\begin{array}{ccc} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ (9x^2yz+yz+\cos(x+y+z)) & (3x^3z+xz+\cos(x+y+z)) & (3x^3y+xy+\cos(x+y+z)) \end{array}$$

$\text{Curl}(F) = i((3x^2+x+\cos(x+y+z)) - (3x^2+x+\cos(x+y+z))) - j((9x^2y + y+\cos(x+y+z)) - (9x^2y+y+\cos(x+y+z))) + k((9x^2z + z + \cos(x+y+z)) - (9x^2z+z+\cos(x+y+z))) = \langle 0,0,0\rangle$. Therefore F is a conservative vector field.

(ii) $f_x = 9x^2yz + yz + \cos(x+y+z)$, $f = 3x^3yz + xyz + \cos(x+y+z) + f(y,z)$

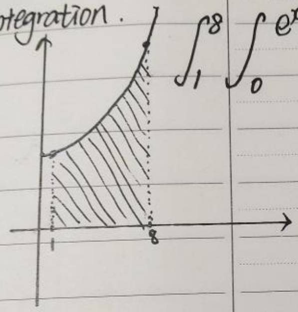
$f_y = 3x^3z + xz + \cos(x+y+z)$, $f = 3x^3yz + xyz + \cos(x+y+z) + f(z)$

$f_z = 3x^3y + xy + \cos(x+y+z)$, $f = 3x^3yz + xyz + \cos(x+y+z)$

(iii) $r(0) = \langle 0, 10, 0\rangle$ $r(\pi) = \langle 0, 8, 0\rangle$

$\text{Int}(F \cdot dr) = f(\pi) - f(0) = 0.694$

3. Sketch the region of integration and change the order of integration. $\int_1^8 \int_0^{e^x+5} F(x,y)\,dy\,dx$



$y = e^x+5$

$x = \ln(y-5)$

$e^1+5 = e+5$, $e^8+5 = e^8+5$

The type II description:

$$\int_0^{e+5}\int_1^8 F(x,y)\,dx\,dy + \int_{e+5}^{e^8+5}\int_{\ln(y-5)}^{8} F(x,y)\,dx\,dy.$$

4. Use Largange multipliers (no credit for other methods) to find the smallest value that $x+y+z$ can be, given that $xyz=8$

$f_x=1$, $f_y=1$, $f_z=1$

$g_x=yz$, $g_y=xz$, $g_z=xy$

$\langle 1,1,1\rangle = \lambda\langle yz, xz, xy\rangle$

$\lambda yz=1$, $\lambda xz=1$, $\lambda xy=1$, $xyz=8$

$\lambda=\frac{1}{4}$, $x=2$, $y=2$, $z=2$

Plug it into f

$f=2+2+2=6$

Ans. 6

5. Compute the volume integral

$$\iiint 7xyz\, dV$$

where E is the region in 3D

$$\{(x,y,z) \mid 5 \le x \le y \le z \le 8\}$$

$$\int_5^8 \int_5^z \int_5^y 7xyz\, dx\, dy\, dz$$

$$= \int_5^8 \int_5^z \frac{7y(y^2-25)z}{2}\, dy\, dz$$

$$= \int_5^8 \frac{7z(z^2-25)^2}{8}\, dz$$

$$= \frac{138411}{16}$$

Ans. $\frac{138411}{16}$

6. By converting to polar coordinates, compute

$$\int_{-7}^{7}\int_{-\sqrt{49-x^2}}^{\sqrt{49-x}} \frac{(x^2+y^2)}{7\pi}\,dy\,dx$$

$x = r\cos(t)$, $y = r\sin(t)$, $\sqrt{(x^2+y^2)} = r$

$$\int_0^{2\pi}\int_0^{7} \frac{r^2}{7\pi}\, r\,dr\,dt$$

$$= \int_0^{2\pi} \frac{343}{4\pi}\,dt = \frac{343}{2}$$

Ans. $\frac{343}{2}$

7. Compute the line integral

$$\int_C \frac{5\sqrt{2}\; x\cdot y\cdot z}{2}\,ds$$

where C is the line-segment joining $(7,6,5)$ and $(3,1,1)$

$C = (7,6,5) + t(-4,-5,-4)$

$r(t) = \langle 7-4t,\ 6-5t,\ 5-4t\rangle$

$r'(t) = \langle -4, -5, -4\rangle$

$|r'(t)| = \sqrt{16+25+16} = \sqrt{57}$

$$\int_C \frac{5\sqrt{2}\,xyz}{2}\,ds = \int_0^1 \frac{5\sqrt{2}(7-4t)(6-5t)(5-4t)}{2}\cdot\sqrt{57}\,dt$$

$$= \frac{705\sqrt{114}}{4}$$

Ans. $\frac{705\sqrt{114}}{4}$

8. Compute $$\int_0^4 \int_{\sqrt[3]{y/4}}^1 e^{x^4}\,dx\,dy$$

$x = \sqrt{y/4}$

$y = 4x^3$

$$\int_0^{4x^3}\int_0^1 e^{x^4}\,dx\,dy$$

$$= \int_0^1\int_0^{4x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^1 4x^3 \cdot e^{x^4}$$

$$= e-1$$

Ans. $e-1$

9. Compute the volume integral

$$\iiint_E \frac{6(x^2+y^2+z^2)}{7\pi}\, dV$$

where

$$E=\{(x,y,z) \mid x^2+y^2+z^2 \le 25\}$$

$$x^2+y^2+z^2=\rho^2$$

$$\iiint_E \frac{6(x^2+y^2+z^2)}{7\pi}\, dV = \int_0^5\int_0^\pi\int_0^{2\pi} \frac{6\rho^2}{7\pi}\rho^2 \sin\phi \, d\theta\, d\phi\, d\rho$$

$$= \int_0^5\int_0^\pi \frac{12\rho^4 \cdot \sin\phi}{7}\, d\phi\, d\rho$$

$$= \int_0^5 \frac{-12\rho^4 \cdot (\cos(5)-1)}{7}\, d\rho$$

$$= \frac{-7500(\cos(5)-1)}{7}$$

Ans. $\frac{-7500(\cos(5)-1)}{7}$

10. Find $\nabla \cdot F$ if

$$F = \langle \sin(2xz), \cos(6yx), \sin(7yz) \rangle$$

$$\nabla \cdot F = \frac{d}{dx} \sin(2xz) + \frac{d}{dy} \cos(6yx) + \frac{d}{dz} \sin(7yz)$$

$$\cancel{= 2z \cos(2xz) +}$$

$$= 2z \cos(2xz) - 6x \sin(6yx) + 7y \cos(7yz)$$

Ans. $2z \cdot \cos(2xz) - 6x \cdot \sin(6yz) + 7y \cdot \cos(7yz)$