Vash Khangura "Quiz" for Lecture 21 Exam Review Section 24

1\. Find the Jacobian of the transformation from $(u,v,w)$-space to $(x,y,z)$-space at the point $(u,v,w) = (12,12,12)$ $u=3, v=3, w=3$

Type: number

$x = uw + v$ $y = uv + w$ $z = vw + u$

$$J = \begin{vmatrix} dx/du = w & dx/dv = 1 & dx/dw = u \\ dy/du = v & dy/dv = u & dy/dw = 1 \\ dz/du = 1 & dz/dv = w & dz/dw = v \end{vmatrix}$$

$= (uv - w)w + (1 - v^2)1 + (vw - u)u = (3\cdot3-3)3 + (1-3^2) + (3\cdot3-3)\cdot3$

$= 6\cdot3 - 8 + 6\cdot3 = 36 - 8 = 28$

2\. (i) Show that F is a conservative vector field

Type: 3D vector of numbers

$F = \langle 4x^3yz + \sin(x+y+z), x^4z + \sin(x+y+z), x^4y + \sin(x+y+z) \rangle$

$$\mathrm{curl}(F) = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ 4x^3yz + \sin(x+y+z) & x^4z + \sin(x+y+z) & x^4y + \sin(x+y+z) \end{vmatrix}$$

$= ((x^4 - \cos(x+y+z)) - (x^4 - \cos(x+y+z)))i + ((4x^3y - \cos(x+y+z)) - (4x^3y - \cos(x+y+z)))j$
$+ ((4x^3z - \cos(x+y+z)) - (4x^3z - \cos(x+y+z)))k$

$= \langle 0,0,0 \rangle$

F is conservative because $\mathrm{curl}(F) = \langle 0,0,0 \rangle$

(ii) Find a function $f(x,y,z)$ such that $F = \nabla f$

Type: function

$\left.\begin{array}{l} f_x = 4x^3yz + \sin(x+y+z) \to f = x^4yz - \cos(x+y+z) \\ f_y = x^4z + \sin(x+y+z) \to f = x^4yz - \cos(x+y+z) \\ f_z = x^4y + \sin(x+y+z) \to f = x^4yz - \cos(x+y+z) \end{array}\right\} f = x^4yz - \cos(x+y+z)$

(iii) Find the line integral $\int_C F\cdot dr$ where C is the curve

Type: number

$r = \langle t, 2t, 3t \rangle$ $0 \le t \le \pi$

$r(0) = (0,0,0)$ $r(\pi) = (\pi, 2\pi, 3\pi)$

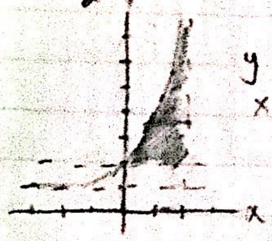
$\int_C F\cdot dr = f(\pi, 2\pi, 3\pi) - f(0,0,0) = (6\pi^6 - \cos(6\pi)) - (0 - \cos(0))$

$= 6\pi^6 - 1 + 1 = 6\pi^6$

Type: Integral + Sketch

3. Sketch the region of integration and change the order of integration

$\int_0^2 \int_2^{e^x+1} F(x,y)\,dy\,dx$



$y = e^x + 1$

$x = \ln(y-1)$

$x: (0,2) \to (\ln(y-1), 2)$

$y: (2, e^x+1) \to (2, e^2+1)$

$\int_{e+1}^{e^2+1} \int_{\ln(y-1)}^{2} F(x,y)\,dx\,dy + \int_2^{e+1}\int_0^2 F(x,y)\,dx\,dy$

Type: Number

4. Use Lagrange multipliers (no credit for other methods) to find the smallest value that $x+y+z$ can be, given that $xyz = 125$

$\nabla f(x,y,z) = \langle 1, 1, 1\rangle$ $\quad\quad$ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\nabla g(x,y,z) = \langle yz, xz, xy\rangle$ $\quad\quad$ $\langle 1,1,1\rangle = \lambda\langle yz, xz, xy\rangle$

$1 = \lambda yz$ $\quad\quad$ $1 = \lambda xz$ $\quad\quad$ $1 = \lambda xy$

$\lambda = 1/yz$ $\quad\quad$ $1 = xz/yz$ $\quad\quad$ $1 = xy/yz$

$1 = x/y$ $\quad\quad$ $z = x$

$y = x$

$g(x,y,z) = xyz = 125$

$= x\cdot x\cdot x = 125 \Rightarrow x^3 = 125$ $\quad$ $x = 5,\ y = 5,\ z = 5$

$x + y + z \Rightarrow 5 + 5 + 5 = 15$

Type: Number

5. Compute the volume integral where $\mathcal{E}$ is the region in 3D $\{(x,y,z) \mid 0 \le x \le y \le z \le 2$

$\iiint_{\mathcal{E}} 12xyz\,dV$ $\quad\quad$ $0 \le z \le 2,\ 0 \le y \le z,\ 0 \le x \le y$

$\int_0^2\int_0^z\int_0^y 12xyz\,dx\,dy\,dz = \int_0^2\int_0^z 6x^2yz \Big|_0^y dy\,dz = \int_0^2\int_0^z 6y^3z\,dy\,dz$

$= \int_0^2 \frac{6}{4}y^4z\Big|_0^z = \int_0^2 \frac{3}{2}z^5 = \frac{1}{4}z^6\Big|_0^2 = 16$

6. By converting to polar coordinates, compute

Type: Number

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{(x^2+y^2)^2}{243\pi}\, dy\, dx = \int_0^{\pi}\int_0^{2} \frac{r^4}{243\pi}\, dr\, d\theta = \int_0^{\pi} \frac{r^5}{1215\pi}\Big|_0^2\, d\theta =$$

$$\int_0^{\pi} \frac{32}{1215\pi}\, d\theta = \frac{32}{1215\pi}\theta\Big|_0^{\pi} = \frac{32}{1215}$$

7. Compute the line integral where $C$ is the line-segment joining $(0,0,0)$ and $(1,1,1)$

Type: Number

$$\int_C \frac{xyz}{5}\, ds \qquad r = \langle t, t, t\rangle \qquad 0 \le t \le 1$$

$$r' = \langle 1, 1, 1\rangle$$

$$\int_0^1 \frac{t^3}{5}\sqrt{1^2+1^2+1^2}\, dt = \frac{\sqrt{3}}{5}\cdot\frac{t^4}{4}\Big|_0^1 = \frac{\sqrt{3}}{20}$$

8. $$\int_0^4\int_{\sqrt[3]{y/4}}^{1} e^{x^4}\, dx\, dy = \int_0^1\int_0^{4x^3} e^{x^4}\, dy\, dx = \int_0^1 ye^{x^4}\Big|_0^{4x^3} dx = \int_0^1 4x^3 e^{x^4}\, dx$$

Type: Number

$$= e^{x^4}\Big|_0^1 = e - 1$$

10. Find $\nabla\cdot F$ if $F = \langle \sin(xz), \sin(xy), \sin(yz)\rangle$

Type: function

$$\operatorname{div} F = \frac{d}{dx}(\sin(xz)) + \frac{d}{dy}(\sin(xy)) + \frac{d}{dz}(\sin(yz))$$

$$= z\cos(xz) + x\cos(xy) + y\cos(yz)$$

9. Compute the volume integral where $\mathcal{E} = \{(x,y,z) \mid x^2+y^2+z^2 \le 4\}$

Type: Number

$$\iiint_{\mathcal{E}} \frac{3}{4\pi}\underbrace{(x^2+y^2+z^2)}_{\rho^2}\, dV \qquad \{(\rho,\phi,\theta) \mid 0 \le \rho \le 2,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

$$\frac{3}{4\pi}\int_0^2\int_0^{\pi}\int_0^{2\pi} \rho^2\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho = \frac{3}{4\pi}\int_0^2\int_0^{\pi} \rho^4\sin\phi\cdot\theta\Big|_0^{2\pi} d\phi\, d\rho = \frac{3}{2}\int_0^2\int_0^{\pi} \rho^4 \sin\phi\, d\phi\, d\rho$$

$$= \frac{3}{2}\int_0^2 -\rho^4\cos\phi\Big|_0^{\pi} d\rho = 3\int_0^2 \rho^4\, d\rho = \frac{3}{5}\rho^5\Big|_0^2 = \frac{96}{5}$$