Lecture 21 attendance quiz Shaun Goda

January February March April May June July August September October November December

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

1) Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space

$$x = uv + w, \quad y = uw + v, \quad z = vw + u \quad \text{at point } (u, v, w) = (4,4,4)$$

type of answer will be a number

$$\begin{vmatrix} v\,i & u\,j & 1\,k \\ w & 1 & u \\ 1 & w & v \end{vmatrix} = \begin{vmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 1 & 4 & 4 \end{vmatrix} = 4(4-4) - 4(16-4) + 1(16-1)$$

$$= 0 - 4(12) + 15$$

$$= -33$$

2) Show that

i. $F = \langle 6x^2yz + 2yz + 2\cos(x+y+z), 2x^3z + 2xz + 2\cos(x+y+z), 2x^3y + 2xy + 2\cos\cdots \rangle$

is a conservative vector

$$P_y = 6x^2z + 2z - 2\sin(x+y+z)$$
$$q_x = 6x^2z + 2z - 2\sin(x+y+z) \Rightarrow P_y = q_x$$

$$P_z = 6x^2y + 2y - 2\cos(x+y+z)$$
$$R_x = 6x^2y + 2y - 2\cos(x+y+z) \Rightarrow P_z = R_x$$

$$q_z = -\sin(x+y+z)$$
$$R_y = -\sin(x+y+z) \Rightarrow q_z = R_y$$

Since $R_y = q_x$, $P_z = R_x$, and $q_z = R_y$, F is consenative.

ii. Find a function $f(x, y, z)$ such that $F = \nabla f$

$$f_x = 6x^2yz + 2yz + 2\cos(x+y+z)$$
$$f_y = 2x^3z + 2xz + 2\cos(x+y+z)$$
$$f_z = 2x^3y + 2xy + 2\cos(x+y+z)$$

$f = \int f_x \, dx = \int (6x^2yz + 2yz + 2\cos(x+y+z))\, dx$

$= 2x^3yz + 2yzx + 2\sin(x+y+z) + h(y,z)$

$\frac{\partial}{\partial y}\left(2x^3yz + 2yzx + 2\sin(x+y+z) + h(y,z)\right)$

$= 2x^3z + 2zx + 2\cos(x+y+z) + h_y(y,z) = f_y$

$h_y(y,z) = 0$, meaning $h(y,z) = g(z)$

$\frac{\partial}{\partial z}\left(2x^3yz + 2yzx + 2\sin(x+y+z) + g(z)\right)$

$= 2x^3y + 2yx + 2\cos(x+y+z) + g_z(z) = f_z$

$g_z(z) = 0$.

"magic function" is $f(x,y,z) = 2x^3yz + 2yzx + 2\sin(x+y+z)$

iii Find the line-integral $\int_C F\, dr$ where C is the curve $r = \langle \sin t, \cos t + 1, \sin 2t \rangle$, $0 \le t \le \pi$

$\int_C F\, dr = \int_0 F(r(t)) \cdot r'(t)\, dt$

$r(0) = \langle 0, 2, 0 \rangle$ & $r(\pi) = \langle 0, 0, 0 \rangle$

using function from ii.

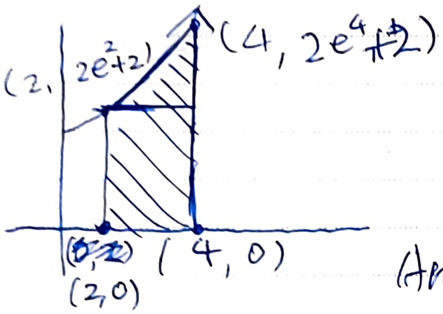
$\int_C F\, dr = f(r(b)) - f(r(a))\ (2(0) + 2(0) + 2(0)) - (2(0) + 2(0) + 2\sin(2))$

$= -2\sin 2$

3) Sketch the area of integration and change the order of integration.

$$\int_2^4 \int_0^{2e^x+2} F(x,y)\,dy\,dx$$

$$y = 2e^x + 2 \Rightarrow \frac{y-2}{2} = e^x \Rightarrow \ln\left(\tfrac{1}{2}y - 1\right) = x$$



$$2 \le x \le 4$$
$$0 \le y \le 2e^x + 2$$
$$\Downarrow$$

Area of $\begin{matrix} 2 \le x \le 4 \\ 0 \le y \le 2e^2+2 \end{matrix}$ + $\begin{matrix} 2e^2+2 \le y \le 2e^4+2 \\ \ln(\tfrac{1}{2}y - 1) < x \le 4 \end{matrix}$

$$\int_2^4 \int_0^{2e^2+2} F(x,y)\,dy\,dx = \int_0^{2e^2+2} \int_2^4 F(x,y)\,dx\,dy + \int_{2e^2+2}^{2e^4+2} \int_{\ln(\frac{1}{2}y-1)}^{4} F(x,y)\,dx\,dy$$

4) use lagrange multiplier to find the smallest value that $2x+2y+2z$ can be, given $xyz = 2$

$$\nabla f = \langle 2, 2, 2 \rangle$$

when $g = xyz = 2$, $\nabla g = \langle yz, xz, xy \rangle$

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2,2,2 \rangle = \lambda \langle yz, xz, xy \rangle$$

$$2 = \lambda yz, \quad 2 = \lambda xz, \quad 2 = \lambda xy$$

$$2x = \lambda xyz \quad 2y = \lambda xyz, \quad 2z = \lambda xyz$$

$$2x = \lambda 2 \quad 2y = \lambda 2 \quad 2z = \lambda 2$$

$\lambda = x \quad \lambda = y \quad \lambda = z$ minimum @ $(1, 1, 1)$

answer is $2(1) + 2(1) + 2(1) = 6$

5) Calculate the volume integral

$$\iiint 96xyz\,dV$$

where E is the region in 3D $\{(x,y,z) \mid 0 \le x \le y \le z \le 2\}$

region in 3D ⇒ $\{0 \le z \le 2,\ 0 \le y \le z,\ 0 \le x \le y\}$

$$\int_0^2 \int_0^z \int_0^y 96xyz\,dx\,dy\,dz$$

$$\int_0^y 96xyz\,dx = \left|48x^2yz\right|_0^y = 48y^3z$$

$$\int_0^z 48y^3z\,dy = \left|12y^4z\right|_0^z = 12z^5$$

$$\int_0^2 12z^5\,dz = \left|2z^6\right|_0^2 = 128$$

6) By converting to polar coordinates, compute $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{(x^2+y^2)^2}{243\pi}\,dy\,dx$

$$\int_0^{2\pi}\int_0^3 \frac{r^5}{243\pi}\,dr\,d\theta = \frac{1}{243\pi}\left|\frac{r^6}{6}\right|_0^3 = \left(\frac{243}{2}\right)\frac{1}{243\pi} = \frac{1}{2\pi}$$

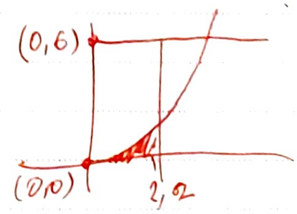
$$\int_0^{2\pi} \frac{1}{2\pi}\,d\theta = \left|\frac{1}{2\pi}\theta\right|_0^{2\pi} = 1$$

7) Compute line integral $\displaystyle\int_C \frac{4\sqrt{3}\,xyz}{3}\,ds$ joining $(0,0,0)$ and $(2,2,2)$.

$\langle 0,0,0\rangle + t(\langle 2,2,2\rangle - \langle 0,0,0\rangle) = \langle 2t, 2t, 2t\rangle$

$$\frac{4\sqrt{3}}{3}\int_0^2 8t^3\sqrt{2^2+2^2+2^2}\,dt = 64\int_0^2 t^3\,dt = 64\left|\frac{t^4}{4}\right|_0^2 = 256$$

8) Compute $\int_0^6 \int_{2\sqrt{y/3}}^{2} 2e^{x^3}\,dx\,dy$ I was not able to solve this one...



$x = 2\sqrt{y/3}$

$\frac{x}{2} = \sqrt{y/3}$

$\frac{x^2}{4} = \frac{y}{3}$

$\frac{3x^2}{4} = y$

Type II description $0 \le y \le 6$

$2\sqrt{y/3} \le x \le 2$

$0 \le y \le \frac{3x^2}{4}$

$0 \le x \le 2$

$$\int_0^2 \int_0^{\frac{3x^2}{4}} 2e^{x^3}\,dy\,dx \Rightarrow \int_0^{\frac{3x^2}{4}} 2e^{x^3}\,dy = \left| 2e^{x^3} y \right|_0^{\frac{3x^2}{4}} = \frac{3\cancel{2}e^{x^3}x^2}{2}$$

$$= \frac{3}{2}\int_0^2 e^{x^3}x^2\,dx = \frac{3}{2}\left| \frac{e^{x^3}}{3} \right|_0^2 = \frac{3}{2}\cdot\frac{e^8}{3} = \frac{e^8}{2} - \frac{1}{3}$$

9) Compute the volume Integral $\iiint_E \frac{5(x^4+y^4+z^4)}{2\pi}\,dV$

where $E = \{(x,y,z) \mid x^4+y^4+z^4 \le 1\}$

$$\frac{5}{2\pi}\int_0^1\int_0^{\pi}\int_0^{2\pi} \rho^8 \sin\phi \,d\theta\,d\phi\,d\rho$$

$$= \frac{5}{2\pi}\left(\int_0^1 \rho^8\,d\rho \cdot \int_0^{\pi} \sin\phi\,d\phi \cdot \int_0^{2\pi} d\theta \right)$$

$$= \frac{5}{2\pi}\left(\left| \frac{\rho^9}{9} \right|_0^1 \cdot \left| -\cos\phi \right|_0^{\pi} \cdot \left| \theta \right|_0^{2\pi} \right) = \frac{5}{2\pi}\left(\frac{1}{9} \cdot 1 \cdot 2\pi \right)$$

$$= \frac{5}{2\pi}\left(\frac{2\pi}{9} \right) = \frac{10\pi}{18} = \frac{5\pi}{9}$$

(10) Find $\nabla \cdot F$ if $F = \langle \sin(2xy), \sin(2yz), \sin(2xz) \rangle$

$\nabla F = 2\cos(2xy) + 2\cos(2yz) + 2\cos(2xz)$