1. Find the Jacobian of the transformation (u, v, w) - space to (x, y, z) - space

$$x = vw + u^2, \quad y = uw + v^2, \quad z = uv + w^2,$$

at the point $(u, v, w) = (1, 2, 4)$

The type of the answer is: NUMBER

$$\det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} 2u & w & v \\ w & 2v & u \\ v & u & 2w \end{pmatrix} = \det \begin{pmatrix} 2 & 4 & 2 \\ 4 & 4 & 1 \\ 2 & 1 & 8 \end{pmatrix}$$

$= 2((4)(8) - (1)(1)) - 4((4)(8) - (1)(2)) + 2((4)(1) - (4)(2))$

$= 2(32 - 1) - 4(32 - 2) + 2(4 - 8)$

$= 2(31) - 4(30) + 2(-4)$

$= 62 - 120 - 8$

$= \boxed{-66}$

i. Show that $F = \langle z^2 + 2xy, x^2 + z, 2xz - 1 \rangle$ is a conservative vector field

ii. Find a function $f(x, y, z)$ such that $F = \nabla f$

iii. Find the line integral $\int_C F \cdot dr$ where C is the curve $r = \langle \cos t, \sin t, \sin 2t \rangle$, $0 \le t \le \pi$

The types of the answer is: For (ii) Multivariable Function For (iii) Number

i.

$$\det \begin{pmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ z^2 + 2xy & x^2 + z & 2xz - 1 \end{pmatrix} = i\left(d/dy(2xz - 1) - d/dz(x^2 + z)\right)$$

$$- j\left(d/dx(2xz - 1) - d/dz(z^2 + 2xy)\right)$$

$$+ k\left(d/dx(x^2 + z) - d/dy(z^2 + 2xy)\right)$$

$$= i(0 - 0) - j(2z - 2z) + k(2x - 2x) = 0i + 0j + 0k = \langle 0, 0, 0 \rangle$$

ii. $f_x = z^2 + 2xy \quad f_y = x^2 + z \quad f_z = 2xz - 1$

$f = \int (z^2 + 2xy)\,dx = z^2 x + x^2 y + g(y, z)$

$f_y = x^2 + z \rightarrow x^2 + g_y(y, z) = x^2 + z$

$f_z = 2xz - 1 \rightarrow 2xz + h_z(z) = 2xz - 1$

$\boxed{f = z^2 x + x^2 y + zy - z}$

iii. $r(0) = \langle 1, 0, 0 \rangle \quad r(\pi) = \langle -1, 0, 0 \rangle$

$\int_C F \cdot dr = f(-1, 0, 0) - f(1, 0, 0) = \boxed{0}$

3. Sketch the region of integration and change the order of integration

$$\int_1^5 \int_0^{e^{2x}} F(x,y)\, dy\, dx$$

The type of the answer is: Sum of two abstract double-integrals

Type 1 description: $\{(x,y) \mid 1 \le x \le 5,\ 0 \le y \le e^{2x}\}$

floor: x-axis: $(1,0)$ to $(5,0)$

left wall: $(1,0)$ to $(1,e^2)$

right wall: $(5,0)$ to $(5,e^{10})$

roof: curve e^{2x}

$y = e^{2x}$ $\quad x = \frac{\ln(y)}{2}$

Type 2 description: $\{(x,y) \mid 0 \le y \le e^2,\ 1 \le x \le 5\} \cup \{(x,y) \mid e^2 \le y \le e^{10},\ \frac{\ln(y)}{2} \le x \le 5\}$

$$\int_0^{e^2} \int_1^5 F(x,y)\, dx\, dy + \int_{e^2}^{e^{10}} \int_{\frac{\ln y}{2}}^{5} F(x,y)\, dx\, dy$$

4. Use Lagrange multipliers (no credit for other methods) to find the smallest value that $x^2 + y^2 + z^2$ can be, given that $xyz = 3$

The type of the answer is: NUMBER

$f = x^2 + y^2 + z^2 \qquad g = xyz = 3$

$\nabla f = \langle 2x, 2y, 2z \rangle \qquad \nabla g = \langle yz, xz, xy \rangle$

$\nabla f = \lambda \nabla g \rightarrow \langle 2x, 2y, 2z \rangle = \lambda \langle yz, xz, xy \rangle$

$2x = \lambda yz \qquad 2y = \lambda xz \qquad 2z = \lambda xy$

$2x^2 = \lambda xyz \qquad 2y^2 = \lambda xyz \qquad 2z^2 = \lambda xyz$

$2x^2 = 3\lambda \qquad 2y^2 = 3\lambda \qquad 2z^2 = 3\lambda$

$8x^2y^2z^2 = 27\lambda^3 \rightarrow 8(xyz)^2 = 27\lambda^3 \rightarrow 8(3)^2 = 27\lambda^3 \rightarrow \frac{8}{3} = \lambda^3 \rightarrow \lambda = \frac{2}{\sqrt[3]{3}}$

$x = \frac{\sqrt{3}}{\sqrt[3]{3}} \qquad y = \frac{\sqrt{3}}{\sqrt[3]{3}} \qquad z = \frac{\sqrt{3}}{\sqrt[3]{3}}$

$f(x,y,z) = \left(\frac{\sqrt{3}}{\sqrt[3]{3}}\right)^2 + \left(\frac{\sqrt{3}}{\sqrt[3]{3}}\right)^2 + \left(\frac{\sqrt{3}}{\sqrt[3]{3}}\right)^2 = \boxed{\frac{9}{\sqrt[3]{3}}}$

5. Compute the volume integral

$$\iiint_E 15xyz\,dV$$

where E is the region in 3D

$$\{(x,y,z) \mid 0 \le x \le y \le z \le 6\}$$

The type of the answer is: NUMBER

$$\{(x,y,z) \mid 0 \le z \le 6,\ 0 \le y \le z,\ 0 \le x \le y\}$$

Iterated integral: $\int_0^1 \int_0^z \int_0^y (15xyz)\,dx\,dy\,dz$

$$= \frac{15x^2yz}{2}\Big|_0^y = \frac{15y^3z}{2}$$

$$= \frac{15y^4z}{8}\Big|_0^z = \frac{15z^5}{8}$$

$$= \frac{15z^6}{48}\Big|_0^1 = \frac{15}{48} = \boxed{\frac{5}{16}}$$

6. By converting to polar coordinates, compute $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{(x^2+y^2)^3}{64\pi}\,dy\,dx$

The type of the answer is: NUMBER

$$\{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$\frac{r^6}{64\pi} \rightarrow \int_0^{2\pi}\int_0^2 \frac{r^7}{64\pi}\,dr\,d\theta = \frac{1}{64\pi}\int_0^{2\pi}\int_0^2 r^7\,dr\,d\theta = \frac{1}{64\pi}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 r^7\,dr\right) = \frac{1}{64\pi}(2\pi)(2^5) = \boxed{1}$$

7. Compute the line integral

$$\int_C \frac{x^2yz}{4}\,ds\,,$$

where C is the line segment joining $(0,0,0)$ and $(1,2,3)$

The type of the answer is: NUMBER

parametric equation of X: $\langle 0,0,0\rangle + t(\langle 1,2,3\rangle - \langle 0,0,0\rangle) = \langle t, 2t, 3t\rangle \quad 0 \le t \le 3$

$r(t) = \langle t, 2t, 3t\rangle \rightarrow r'(t) = \langle 1,2,3\rangle$

$\|r'(t)\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$$\int_0^3 \frac{(t^2)(2t)(3t)}{4}(\sqrt{14})\,dt = \int_0^3 \frac{(3\sqrt{14})(t^4)}{2}\,dt = \frac{(3\sqrt{14})(t^5)}{10}\Big|_0^3 = \boxed{\frac{729\sqrt{14}}{10}}$$

8. Compute $\int_0^8 \int_{y/2}^2 e^{x^2}\,dx\,dy$

The type of the answer is: NUMBER

$\{(x,y) \mid 0 \le y \le 8,\ y/2 \le x \le 2\}$

$\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2x\}$

$\int_0^2 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^2 (2x)(e^{x^2})\,dx = \int_0^4 e^u\,du = e^u\Big|_0^4 = \boxed{e^4 - 1}$

$u = x^2 \quad du = 2x\,dx$

9. Compute the volume integral $\iiint_E \frac{2(x^2+y^2+z^2)}{3\pi}\,dV$ where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 3\}$

The type of the answer is: NUMBER

$\{(\rho,\phi,\theta) \mid 0 \le \rho \le 3,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

$\frac{2}{3\pi}\int_0^3\int_0^\pi\int_0^{2\pi} \rho^2\rho^2 \sin\phi\,d\theta\,d\phi\,d\rho = \frac{2}{3\pi}\left(\int_0^3 \rho^4\,d\rho\right)\left(\int_0^\pi \sin\phi\,d\phi\right)\left(\int_0^{2\pi} d\theta\right)$

$= \frac{2}{3\pi}\left(\frac{\rho^5}{5}\Big|_0^3\right)\left(-\cos\phi\Big|_0^\pi\right)\left(\theta\Big|_0^{2\pi}\right) = \frac{2}{3\pi}\left(\frac{243}{5}\right)(2)(2\pi) = \boxed{\frac{648}{5}}$

10. Find $\nabla \cdot F$ if $F = \langle \cos(x^2z), \cos(y^2x), \cos(z^2y) \rangle$

The type of the answer is: Multivariable Function

$\nabla \cdot F = \frac{d}{dx}(\cos(x^2z)) + \frac{d}{dy}(\cos(y^2x)) + \frac{d}{dz}(\cos(z^2y))$

$= \boxed{-2xz\sin(x^2z) - 2yx\sin(y^2x) - 2zy\sin(z^2y)}$