2017 Exam (My own Problems)

1) Q: Find the Jacobian of the transformation from (u,v,w)-space to (x,y,z)-space

$x = uv+w$, $y = uw+v$, $z = vw+u$

at the point $(u,v,w)=(4,4,4)$

A: $$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} v & u & 1 \\ w & 1 & u \\ 1 & w & v \end{vmatrix}$$

$$= v(v-uw) - u(wv-u) + 1(w^2-1)$$

$$= 4(4-16) - 4(16-4) + 1(16-1)$$

$$= -48 - 48 + 15 = \boxed{-81}$$

2) Q: (i) Show that $F = \langle 3x^2yz+yz+\cos(x+y+z),\ x^3z+xz+\cos(x+y+z),\ x^3y+xy+\cos(x+y+z)\rangle$

is a conservative vector field.

(ii) Find a function $f(x,y,z)$ such that $F = \nabla f$

(iii) Find the line-integral $\int_C F\cdot dr$ where C is the curve

$r = \langle \sin t,\ \cos t + 1,\ \sin 2t\rangle$, $\frac{\pi}{2} \le t \le \pi$

A:

(i) Show curl $F = 0$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz+yz+\cos(x+y+z) & x^3z+xz+\cos(x+y+z) & x^3y+xy+\cos(x+y+z) \end{vmatrix}$$

$$= i\left((x^3+x+\sin(x+y+z)) - (x^3+x+\sin(x+y+z))\right) - j\left((3x^2y+y+\sin(x+y+z)) - (3x^2y+y+\sin(x+y+z))\right)$$
$$+ \hat{k}\left((3x^2z+z+\sin(x+y+z)) - (3x^2z+z+\sin(x+y+z))\right)$$

$= 0 \rightarrow$ conservative

(ii) $f_x = 3x^2yz+yz+\cos(x+y+z)$

$$\int 3x^2yz+yz+\cos(x+y+z)\,dx = x^3yz+xyz+\sin(x+y+z)$$

$$f = x^3yz+xyz+\sin(x+y+z)+g(y)$$

$$\frac{\partial}{\partial y}\left(x^3yz+xyz+\sin(x+y+z)+g(y)\right) = x^3z+xz+\cos(x+y+z)+g'(y) = x^3z+xz+\cos(x+y+z)$$

$$g'(y)=0$$

$$g(y)=h(z)$$

$$\frac{\partial}{\partial z}\left(x^3yz + xyz + \sin(x+y+z) + h(z)\right)$$

$$= x^3y + xy + \cos(x+y+z) + h'(z) = x^3y + xy + \cos(x+y+z)$$

$$h'(z) = 0$$

$$h(z) = C$$

so $\boxed{f = x^3yz + xyz + \sin(x+y+z)} + C$

(iii) Fundamental theorem of line integrals

f(end point) - f(start point)

$r(\frac{\pi}{2}) = \langle 1, 1, 0 \rangle$

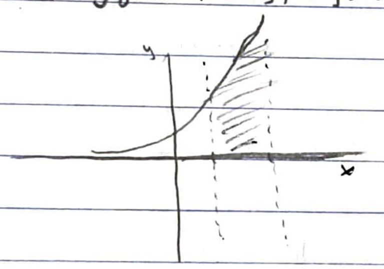
$r(\pi) = \langle 0, 0, 0 \rangle$

$\int_C F\,dr = f(0,0,0) - f(1,1,0) = (0) - (\sin(2)) = \boxed{-\sin(2)}$

3) Q: Sketch the region of integration and change the order of integration

$$\int_1^2 \int_0^{e^x+2} F(x,y)\,dy\,dx$$

A:



$\{(x,y) \mid 1 \le y \le e+2,\ 1 \le x \le 2\} \cup \{(x,y) \mid e+2 \le y \le e^2+2,\ \ln(y-2) \le x \le 2)$

$$\int_0^{e+2} \int_1^2 F(x,y)\,dx\,dy + \int_{e+2}^{e^2+2} \int_{\ln(y-2)}^{2} F(x,y)\,dx\,dy$$

4) Q: Use Lagrange Multipliers to find the smallest value that x+y+z can be, given that

$xyz = 2$

A: $\nabla f(x,y,z) = \langle 1, 1, 1 \rangle$

$\nabla g(x,y,z) = \langle yz, xz, xy \rangle$

$\nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z)$

$1 = \lambda yz, \quad 1 = \lambda xz, \quad 1 = \lambda xy$

$\frac{y}{x} = 1 \qquad x = y, \quad \frac{z}{y} = 1, \quad y = z$

$x = y = z$

$x^3 = 2, \quad x = \sqrt[3]{2}$

$f(\sqrt[3]{2}, \sqrt[3]{2}, \sqrt[3]{2}) = \boxed{3(\sqrt[3]{2})}$

5) Q: Compute the Volume Integral

$$\iiint_E 48\,xyz\,dV$$

where E is the region in 3D

$$\{(x,y,z) \mid 0 \le x \le y \le z \le 2\}$$

A: $\{(x,y,z) \mid 0 \le z \le 2,\ 0 \le y \le z,\ 0 \le x \le y\}$

$$\int_0^2 \int_0^z \int_0^y 48xyz\,dx\,dy\,dz$$

$= \int_0^y 48xyz\,dx = (24x^2yz)|_0^y = 24y^3z$

$= \int_0^z 24y^3z\,dy = (6y^4z)|_0^z = 6z^5$

$= \int_0^2 6z^5\,dz = (z^6)|_0^2 = \boxed{64}$

6) Q: $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{(x^2+y^2)^2}{16\pi}\,dy\,dx$ convert to polar coordinates and compute.

A: $x = r\cos\theta$ $\quad r=2$, so $0 \le r \le 2$

$y = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$r^2 = x^2+y^2$

$$\int_0^{\pi} \int_0^2 \frac{(r^2)^2}{16\pi}\,r\,dr\,d\theta$$

$= \frac{1}{16\pi} \int_0^{\pi} \int_0^2 r^5\,dr\,d\theta$

$= \int_0^2 r^5\,dr = \left(\frac{r^6}{6}\right)\Big|_0^2 = \frac{64}{6}$

$= \int_0^{2\pi} \frac{64}{6}\,d\theta = \left(\frac{64}{6}\theta\right)\Big|_0^{2\pi} = \frac{128\pi}{6}\left(\frac{1}{16\pi}\right) = \frac{8}{6} = \boxed{\frac{4}{3}}$

7) Q: Compute the line integral $\int_C \frac{4\sqrt{3}\,xyz}{3}\,ds$, where C is the line segment joining (0,0,0) and (2,2,2)

A: Parametric equation: $\langle 0,0,0\rangle + t\langle 2,2,2\rangle = \langle 2t,2t,2t\rangle$, $0 \le t \le 1$

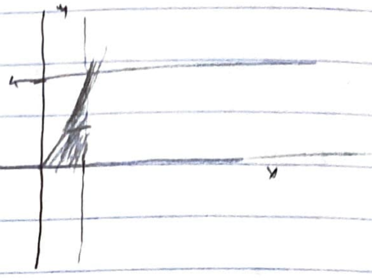
$x = 2t,\ y = 2t,\ z = 2t$, $\quad r(t) = \langle 2t,2t,2t\rangle$

$r'(t) = \langle 2,2,2\rangle$

$\|r'(t)\| = \sqrt{12} = 2\sqrt{3}$

$\int_1^1 \frac{4\sqrt{3}(2t)(2t)(2t)}{3}\,2\sqrt{3}\,dt = \int_0^1 64t^3\,dt = (16t^4)|_0^1 = \boxed{16}$

8) Q: Compute $\int_0^4 \int_{\sqrt{y/4}}^1 e^{x^4}\,dx\,dy$



A: $y=4x^2$

$$\int_0^1 \int_0^{4x^2} e^{x^4}\,dy\,dx$$

$$= \int_0^{4x^2} e^{x^4}\,dy = (ye^{x^4})\Big|_0^{4x^2} = 4x^2e^{x^4}$$

$$= \int_0^1 4x^2e^{x^4}\,dx$$

$u: x^4$

$du: 4x^2dx$

$dx = \frac{du}{4x^2}$

$u(0): 0^4 = 0$

$u(1): 1^4 = 1$

$$\int_0^1 e^u\,du = e^1 - e^0 = \boxed{e}$$

9) Q: Compute the volume integral

$$\iiint_E \frac{10(x^2+y^2+z^2)}{4\pi}\,dV$$

$E: \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

A: spherical coordinates

$$\{(\rho,\phi,\theta) \mid 0 \le \rho \le 1,\ 0 \le \phi, \pi,\ 0 \le \theta, \cancel{2\pi}\}$$

$x^2+y^2+z^2 = \rho^2$, $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$\frac{10}{4\pi}\int_0^1\int_0^\pi\int_0^{2\pi} \rho^2\rho^2\sin\phi\,d\theta\,d\phi\,d\rho = \frac{10}{4\pi}\int_0^1\int_0^\pi\int_0^{2\pi} \rho^4\sin\phi\,d\theta\,d\phi\,d\rho$$

$$\frac{10}{4\pi}\left(\int_0^1 \rho^4\,d\rho\right)\cdot\int_0^\pi(\sin\phi\,d\phi)\cdot\int_0^{2\pi} d\theta$$

$$= \frac{10}{4\pi}\left(\frac{\rho^5}{5}\Big|_0^1\right)\left(-\cos\phi\Big|_0^\pi\right)(2\pi)$$

$$= \frac{1}{2\pi}(2)(2\pi) = \boxed{2}$$

10) Q. Find $\nabla\cdot F$ if $F = \langle \sin(xy), \sin(yz), \sin(xz)\rangle$

A: $\text{Div}\,F = \frac{\partial}{\partial x}(\sin(xy)) + \frac{\partial}{\partial y}(\sin(yz)), \frac{\partial}{\partial z}(\sin(xz))$

$$= \boxed{y\cos(xy) + z\cos(yz) + x\cos(xz)}$$