q21 Rahul Poleja

2017 → Practice Exam → Make Up Similar Problems and Solve:

Do Problems and Add Types

① Find the jacobian of the transformation from (u,v,w)-space to (x,y,z)-space

$$x = uv - w \qquad y = vw + u \qquad z = uv + w$$

at the point $(u,v,w) = (1,1,1)$

Type → Number

Answer:

$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} v & u & -1 \\ 1 & w & v \\ v & u & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$1(1-1) - 1(1-1) + 1(1-1)$$

$$= \boxed{0}$$

② (i) Show that

$$F = \langle 3x^2 + \cos(x), \cancel{y}^2 + \cos(y), z + \sin(\cancel{z}) \rangle$$

is a conservative vector field

Answer: Show $\text{curl}(F) = \langle 0,0,0 \rangle$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2+\cos(x) & \cancel{y}^2+\cos(y) & z+\sin(\cancel{z}) \end{vmatrix}$$

$$i(0) - j(0) + k(0)$$

$$= \langle 0,0,0 \rangle$$

(ii) Find a Function $f(x,y,z)$ such that $F = \nabla f$

Type: Function

$F_x = 3x^2 + \cos(x)$; $F_y = y^2 + \cos(y)$; $F_z = z + \sin(y)$

$f = \int (3x^2 + \cos(x))\,dx = x^3 + \sin(x) + G(y,z)$

$G'(y,z) = \int y^2 + \cos(y)\,dy = y^3 + \sin(y)$

$F = \int x^3 + \sin(x) + y^2 + \cos(y)\,dy$

$f = x^3 + \sin(x) + \frac{y^3}{3} + \sin(y) + h(z)$

$\int h'(z) = \int z + \sin(y)\,dz = \frac{z^2}{2} + z\sin(y)$

No +C needed

$f(x,y,z) = x^3 + \sin(x) + \frac{y^3}{3} + \sin(y) + \frac{z^2}{2} + z\sin(y)$

(iii) Find the line integral $\int_C F \cdot dr$ where C is the curve $r = \langle \sin t, \cos t + 1, \sin 2t \rangle$, $0 \le t \le \pi$

Type: Number

Answer: Fundamental Theorem of Line Integrals

$F(\text{end}) - F(\text{start})$

$r(0) = \langle 0, 2, 0 \rangle$ $r(\pi) = \langle 0, 0, 0 \rangle$

$\int_C F\,dr = F(0,0,0) - F(0,2,0)$

$= (0 + 0 + 0 + 0 + 0 + 0) - \left(\frac{8}{3} + \sin(2)\right)$

$= \boxed{-\frac{8}{3} - \sin(2)}$

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(3) Sketch Region of Integration and change order of integration

$$\int_0^2 \int_0^{e^x} F(x,y)\,dy\,dx$$

Type: Integral

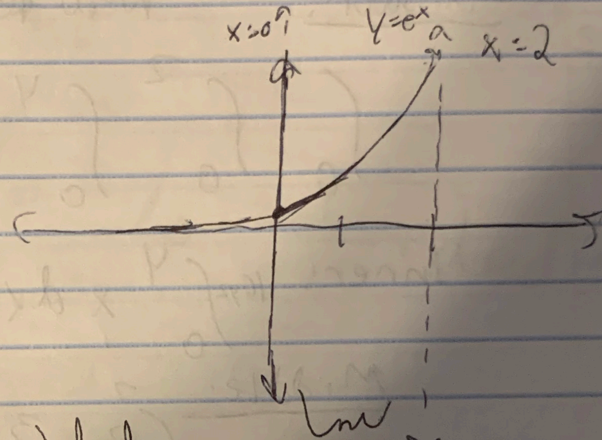
Answer:

y goes from e^0 to e^2

x goes from $y = e^x$ to 2

$x = \ln(y)$

$$\int_{e^0}^{e^2} \int_{\ln(y)}^{2} F(x,y)\,dx\,dy$$



(4) Use Lagrange Multipliers to find the smallest value that $x+y+z$ can be given that $xyz=2$

Type: Number

Answer:

$\nabla f = \langle 1,1,1 \rangle$

$\nabla g = \langle yz, xz, xy \rangle$

$\nabla f = \lambda \nabla g$

$1 = \lambda yz \quad 1 = \lambda xz \quad 1 = \lambda xy$

Set equal:

$\frac{\cancel{\lambda} yz}{\cancel{\lambda z}} = \frac{\cancel{\lambda xz}}{\cancel{\lambda z}} \quad y = x$

$\frac{\lambda x z}{\lambda x} = \frac{\lambda x y}{\lambda x} \quad z = y$

so $x = y = z$

$\sqrt[3]{x^3} = \sqrt[3]{2} \quad x = \sqrt[3]{2} = y = z$

$\sqrt[3]{2} + \sqrt[3]{2} + \sqrt[3]{2} = \boxed{3\sqrt[3]{2}}$

⑤ Compute the volume integral

$$\iiint_E 16xyz\,dV$$

$\{(x,y,z) \mid 0 \le x \le y \le z \le 1\}$

Type: Number

answer: New Region: $\{(x,y,z) \mid 0 \le z \le 1,\ 0 \le y \le z;\ 0 \le x \le y\}$

$$\int_0^1 \int_0^z \int_0^y 16xyz\,dx\,dy\,dz$$

Inner: $16yz\int_0^y x\,dx = \frac{x^2}{2}\Big]_0^y = \frac{y^2}{2}$

Middle:

$$\int_0^z 8y^3z\,dy \to 8z\int_0^z y^3\,dy$$

$$\frac{y^4}{4}\Big]_0^z = \frac{z^4}{4} \cdot 8z = 2z^5$$

Outer:

$$\int_0^1 2z^5\,dz = \frac{z^6}{3}\Big]_0^1 = \boxed{\frac{1}{3}}$$

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(6) Compute

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \frac{(x^2+y^2)}{10\pi}\,dy\,dx$$

Type: Number

$\{(r,\theta) \mid 0 \le r \le 4, 0 \le \theta \le 2\pi\}$

$\sqrt{16-r^2\cos^2\theta} = r\sin\theta$

$$\int_0^{2\pi}\int_0^4 \frac{r}{10\pi}\, r\,dr\,d\theta$$

Inner:

$$\frac{1}{10\pi}\int_0^4 r^2\,dr$$

$$= \frac{r^3}{3}\Big]_0^4 = \frac{64}{3}\cdot\frac{1}{10\pi} = \frac{64}{30\pi}$$

Outer:

$$\frac{64}{30\pi}\int_0^{2\pi} d\theta = 2\pi\left(\frac{64}{\underset{15}{30\pi}}\right)$$

$$= \boxed{\frac{64}{15\pi}}$$

(7) Compute the line integral

$\int_C \frac{2xyz}{5}\,ds$ where C is the line segment joining $(0,0,0)$ + $(1,1,1)$

Type: Number

answer: $\langle 0,0,0\rangle + t\langle 1,1,1\rangle = \overset{r(t)}{\langle t,t,t\rangle}$ $0 \le t \le 1$

$x(t) = t$

$y(t) = t$

$z(t) = t$

$r'(t) = \langle 1,1,1\rangle\,dt$

$|r'(t)| = \sqrt{3}\,dt = ds$

Integral: $\int_0^1 \frac{2 t \cdot t \cdot t}{5} \sqrt{3}\, dt$

$= \frac{2\sqrt{3}}{5} \int_0^1 t^3\, dt = \frac{t^4}{4}\Big]_0^1 = \frac{1}{4} \cdot \frac{2\sqrt{3}}{5}$

$= \frac{2\sqrt{3}}{20} = \boxed{\frac{\sqrt{3}}{10}}$

⑧ Compute

$$\int_1^3 \int_{\sqrt{y/2}}^1 e^{x^4}\, dx\, dy$$

Type: Number

Answer:

x goes from $\sqrt{\frac{1}{2}}$ to 1

y goes from 1 to $2x^2$

$$\int_{\sqrt{1/2}}^1 \int_1^{2x^2} e^{x^4}\, dy\, dx$$

inner:

$$e^{x^4} \int_1^{2x^2} dy = e^{x^4}(2x^2 - 1)$$

outer:

$$\int_{\sqrt{1/2}}^1 2x^2 e^{x^4} - e^{x^4} = 0$$

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⑨ Compute volume integral

$$\iiint_E \frac{x^2+y^2+z^2}{\pi}\,dV$$

where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1$

Type: Number

Turn into spherical coordinates

$x^2+y^2+z^2 = \rho^2$ and $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

volume:

$$\frac{1}{\pi}\int_0^1 \int_0^{\pi} \int_0^{2\pi} \rho^2 \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

$$= \boxed{\pi}$$

⑩. Find $\nabla \cdot F$; F

$F = \langle \sin(xy), \cos(yz), \cos(xz) \rangle$

Type: Function

Answer:

$$\frac{\partial}{\partial x}\sin(xy) + \frac{\partial}{\partial y}\cos(yz) + \frac{\partial}{\partial z}\cos(xz)$$

$$= y\cos(xy) - z\sin(yz) - x\sin(xz)$$