q21 Orion Kress-Sanfilippo

1) Find Jacobian of following transformation

$x = u^2 - wv \quad y = v^2 - uw \quad z = w^2 - vu$

@ $(u, v, w) = (3, 1, 3)$

$$\text{Jac} = \det\begin{vmatrix} 2u & -1 & -1 \\ -1 & 2v & -1 \\ -1 & -1 & 2w \end{vmatrix} = \det\begin{vmatrix} 6 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 6 \end{vmatrix}$$

$$= 6(13) - (+1(+5)) + (-1(1-2)) =$$

$$78 - 5 + 1 = \boxed{74}$$

2) Show that

$F = \langle y^3 z + 3z + e^{x+z}, 3xy^2 z, xy^3 + 3x + e^{x+z} \rangle$ is cons.

$$\text{curl}(F) = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 z + 3z + e^{x+z}; & 3xy^2 z; & xy^3 + 3x + e^{x+z} \end{vmatrix}$$

$$= (3xy^2 - 3xy^2)\hat{i} - ((y^3 + 3 + e^{x+z}) - (y^3 + 3 + e^{x+z}))\hat{j}$$
$$+ (3y^2 z - 3y^2 z)\hat{k} = \boxed{0} \checkmark \text{ Conservative}$$

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2) ii) $F = \langle y^3 z + 3z + e^{x+z},\ 3xy^2 z,\ xy^3 + 3x + e^{x+z} \rangle$

$$\int y^3 z + 3z + e^{x+z}\, dx = xy^3 z + 3xz + e^{x+z}$$

$$F = \nabla f \quad \boxed{f = xy^3 z + 3xz + e^{x+z}}$$

iii) Since $F = \nabla f$, $\int_a^b F \cdot dr = f(b) - f(a)$ $\left(\frac{\text{Path}}{\text{Independent}}\right)$

$$r = \langle \sin(t),\ t^2,\ 1-\cos(t) \rangle \quad 0 \le t \le \frac{\pi}{2}$$

$$f(r(t)) = (\sin(t))(t^6)(1-\cos(t)) + 3(\sin(t))(1-\cos(t)) + e^{\sin(t)+1-\cos(t)}$$

$$f\left(r\left(\frac{\pi}{2}\right)\right) = (1)\left(\frac{\pi}{2}\right)^6(1-0) + 3(1)(1-0) + e^{1+1-0}$$

$$= 3 + \frac{\pi^6}{64} + e^2$$

$$f(r(0)) = (0)(0)^6(1-1) + 3(0)\ldots + e^{0+1-1} =$$

$$f(r(0)) - f\left(r\left(\frac{\pi}{2}\right)\right) = \int_C F \cdot dr = \boxed{-2 - \frac{\pi^6}{64} - e^2}$$

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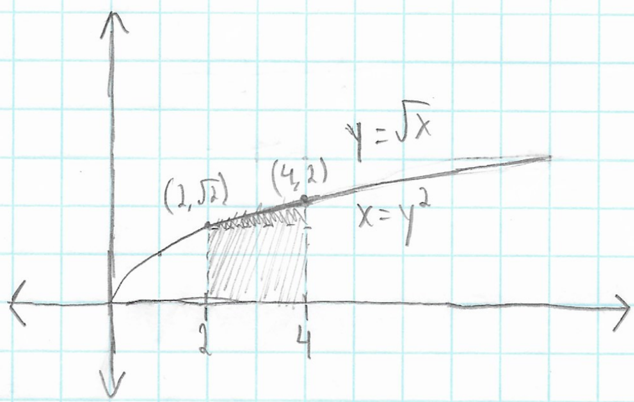
3) Sketch Integration Region & change order of Int.

$$\int_2^4 \int_0^{\sqrt{x}} F(x,y)\,dy\,dx$$



$$\boxed{= \int_0^{\sqrt{2}} \int_2^4 F(x,y)\,dx\,dy + \int_{\sqrt{2}}^{2} \int_{y^2}^{4} F(x,y)\,dx\,dy}$$

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4) Use Lagrange multipliers to find the smallest value that $x+2y-z$ can be, given that $xyz=\sqrt{2}$

$f = x+2y-z \quad g_0 = xyz = \sqrt{2}$

$\nabla f = \langle 1, 2, -1 \rangle \quad \nabla g = \langle yz, xz, xy \rangle \quad \lambda \nabla f = \nabla g$

$\lambda = yz \quad xyz = \sqrt{2}$

$2\lambda y = \cancel{x}z$

$-\lambda = xy$

Solved Using Maple

Ans $= \boxed{f\left(\sqrt{2}, \frac{-\sqrt{2}}{2}, \sqrt{2}\right)} = \sqrt{2} + 2\left(-\frac{\sqrt{2}}{2}\right) + \sqrt{2}$

$\boxed{= -\sqrt{2}}$

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5) Compute the Volume integral

$$\iiint_E 8xyz\,dV$$

$$E: \{(x,y,z) \mid 0 \le y \le x \le z \le 1\}$$

$$= \int_0^1 \int_0^z \int_0^x 8xyz\,dy\,dx\,dz$$

Using Maple

Ans $= \boxed{\frac{1}{6}}$

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6) Using polar coords, compute

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{(x^2+y^2)^{3/2}}{4\pi}\, dy\, dx$$

$0 \leq r \leq 2$

$0 \leq \theta \leq \pi$

$x = r\cos\theta$
$y = r\sin\theta$
$\sqrt{x^2+y^2} = r$

$$= \int_{0}^{\pi}\int_{0}^{2} \frac{(r^2)^{3/2}}{4\pi}\, r\, dr\, d\theta$$

Using Maple

$\boxed{\text{Ans} = \frac{8}{5}}$

7) Compute the line integral

$$\int_C 6\sqrt{2}\, xyz\, dS$$

Where C is the line segment joining (0, 1, 0) & (1, -1, 1)

$r(t) = \langle 0, 1, 0\rangle + t\langle 1, -2, 1\rangle$

Ans

$= \int f(r(t))\, dr =$

Done in Maple

$dr = \sqrt{1^2+(-2)^2+(1)^2} = \sqrt{6}$

$$\int_0^1 6\sqrt{2}(t)(1-2t)(t)(\sqrt{6})\, dt$$

Using Maple

$\boxed{= -2\sqrt{3}}$

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8) Compute $\int_0^2 \int_{\sqrt{y/2}}^1 2e^{x^3}\,dx\,dy$

Region of Int:

$= \int_0^1 \int_0^{2x^2} 2e^{x^3}\,dy\,dx$

$= \int_0^1 4x^2 e^{x^3}\,dx \quad u=e^{x^3} \quad \frac{4}{3}du = (\cancel{x^2} e^{x^3})\frac{4}{\cancel{3}}$

$= \int \frac{4du}{3} = \left(\frac{4}{3}e^{x^3}\right)\Big|_0^1 = \frac{4}{3}(e-1)$

9) Compute Volume Int:

$$\iiint_E \frac{5(x^2+y^2+z^2)}{2\pi}\,dV$$

$E: \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

$0 \le \rho \le 1 \quad 0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi$

$\rho^2 = (x^2+y^2+z^2)$

$\text{Ans} = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \frac{5}{2\pi}(\rho)^2\left(\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi\right)$

Using Maple

$= 1-\cos(\pi) = \boxed{2}$

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10) Find $\nabla \cdot F$ if

$$F = \langle \cos(xy), \cos(yz), \cos(xz) \rangle$$

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} =$$

$$\boxed{-y\sin(xy) - z\sin(yz) - x\sin(xz)}$$