Exam 2 prep

1) find Jacobian of $(2u+v,\ 2v+w,\ 2w+uv)$ @ $(2,2,2)$

$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

• using partial derivatives, solve the determinant

$$\left.\begin{matrix} \frac{\partial(2u+v)}{\partial u} = 2 & \frac{\partial(2u+v)}{\partial v} = 1 & \frac{\partial(2u+v)}{\partial w} = 0 \\ \frac{\partial(2v+w)}{\partial u} = 0 & \frac{\partial(2v+w)}{\partial v} = 2 & \frac{\partial(2v+w)}{\partial w} = 1 \\ \frac{\partial(2w+uv)}{\partial u} = v & \frac{\partial(2w+uv)}{\partial v} = u & \frac{\partial(2w+uv)}{\partial w} = 2 \end{matrix}\right\} \to -2u+v+8$$

• substituting $(2,2,2) \to -2u+v+8 = -2(2)+(2)+8 = \boxed{6}$

double check $2(4-2) - 1(0-2) - 0 = 2(2)+2 = 6$

2) (i) show that $F = \langle 6x^2yz + yz + \cos(2x+2y+2z),\ 2x^3z + xz + \cos(2x+2y+2z),$
$2x^3y + xy + \cos(2x+2y+2z)\rangle$ is conservative.

(ii) find $f(x,y,z)$ so that $F = \nabla f$

(iii) find line integral $\int_C F\,dr$ where C is $r = \langle 2\sin t, 2\cos t + 1, 2\sin 2t\rangle$, $0 \le t \le \pi$

(i) $\text{curl } F = \left(\underset{①}{\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}},\ \underset{②}{\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}},\ \underset{③}{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}\right)$

• all subtraction values must $= 0 \to \langle 0,0,0\rangle$ to be conservative.

$$\left.\begin{matrix} ① \ (2x^3+x-\sin(2x+2y+2z)\cdot 2) - (2x^3+x\cdot\sin(2x+2y+2z)\cdot 2) = 0 \\ ② \ (6x^2y+y-\sin(2x+2y+2z)\cdot 2) - (6x^2y+y-\sin(2x+2y+2z)\cdot 2) = 0 \\ ③ \ (6x^2z+z-\sin(2x+2y+2z)\cdot 2) - (6x^2z+z-\sin(2x+2y+2z)\cdot 2) = 0 \end{matrix}\right\} \to \langle 0,0,0\rangle$$

conservative!

(ii) • b/c ∇f is (f_x, f_y, f_z), our F vals are partial derivatives.

$\int 6x^2yz + yz + \cos(2x+2y+2z)\,dx = xyz(2x^2+1) + \frac{\sin(2x+2y+2z)}{2} + C$

due to $\varphi(x,y,z) = 0$ and $\varphi(y,z) = \psi(z)$, they both $= 0$ and ans. is

$\boxed{xyz(2x^2+1) + \frac{\sin(2x+2y+2z)}{2}}$

(iii) start pt: $(0,3,0)$ and endpt: $(0,-1,0)$ • substitute 0 and π into $\langle 2\sin t, 2\cos t + 1, 2\sin 2t\rangle$

$f(0,3,0) - f(0,-1,0) =$

$\left(0(0+1) + \frac{\sin(0+6+0)}{2}\right) - \left(0(0+1) + \frac{\sin(0+(-2)+0)}{2}\right) = \frac{\sin 6}{2} - \frac{\sin(-2)}{2}$

approx. $\boxed{0.31494}$

3) sketch region of integration and change order of integration

$$\int_2^4 \int_0^{e^{2x+2}} F(x,y)\,dy\,dx$$

· find all boundaries

$[(x,y) = 2 \le x \le 4,\ 0 \le y \le e^{2x+2}]$ → original integration

$x \to (2,0)$ to $(4,0)$

$(2,0)$ to $(2, e^{2x+2})$ and $(4,0)$ to $(4, e^{2x+2})$

or $(2,0)$ to $(2, e^{6})$ and $(4,0)$ to $(4, e^{10})$

top boundary is $y = e^{2x+2} \to \ln(y) = 2x+2 \to \ln(y)-2 = 2x \to x = \frac{\ln(y)-2}{2}$

$$\boxed{[(x,y) = 2 \le y \le e^6,\ 2 \le x \le 4] \cup [(x,y) = e^6 \le y \le e^{10},\ \frac{\ln(y)-2}{2} \le x \le 4]}$$

4.) use largange multipliers to find $2x+2y+2z$, given $\frac{xyz}{2} = 1$

· we set up our f and g

$f = 2x+2y+2z,\quad g = \frac{xyz}{2} - 1$

· find ∇ grad of both

$\nabla f = \langle 2,2,2 \rangle \quad \nabla g = \langle \frac{yz}{2}, \frac{xz}{2}, \frac{xy}{2} \rangle$

· now we find constant multiplier

$\langle 2,2,2 \rangle = \lambda \langle \frac{zy}{2}, \frac{xz}{2}, \frac{xy}{2} \rangle$

$1 = \lambda \frac{yz}{2},\quad 1 = \lambda \frac{xz}{2},\quad 1 = \lambda \frac{xy}{2}$

$2 = \lambda yz,\quad 2 = \lambda xz,\quad 2 = \lambda xy$

$2x = \lambda xyz,\quad 2y = \lambda xyz,\quad 2z = \lambda xyz$

divide by 2 → $x = \lambda \frac{xyz}{2},\ y = \lambda \frac{xyz}{2},\ z = \lambda \frac{xyz}{2}$

$\frac{xyz}{2} = 1 \to \frac{\lambda^3}{2} = 1 \quad x = \lambda = y = z$, so

$f(1,1,1) = (2(2)+2(2)+2(2)) = \boxed{12}$

5.) $\iiint_E 24xyz\,dV$ for $E = [(x,y,z) = 0 \le x \le y \le z \le 4]$

$\int_0^4 \int_0^z \int_0^y 24xyz\,dx\,dy\,dz \to \int_0^y 24xyz\,dx = 12x^2yz \Big|_0^y \to 12y^3z$

$\int_0^z 12y^3z\,dy = 3y^4z \Big|_0^z \to 3z^5 \to \int_0^4 3z^5 \to \frac{z^6}{2}\Big|_0^4 = \boxed{2048}$

6.) convert to polar coordinates & compute: $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \frac{(x^2+y^2)^3}{16\pi}\,dy\,dx$

· We know $r^2 = x^2+y^2$

$\frac{(r^2)^3}{16\pi}$ ~~dxdy~~ → has a radius of 4, 2π full circle

· dydx is now $r\,dr\,d\theta$

$\int_0^{2\pi}\int_0^4 \frac{r^6}{16\pi}\, r\,dr\,d\theta = \frac{1}{16\pi}\int_0^{2\pi}\int_0^4 r^7\,dr\,d\theta = \frac{1}{16\pi}(2\pi)\left(\frac{r^8}{8}\Big|_0^4\right) = \frac{1}{8\cancel{\pi}}\cdot 8192 = \boxed{1024}$

7.) compute line integral $\int_c \frac{8\sqrt{3}\,xyz}{3}\,ds$ where $C = (0,0,0)$ and $(2,2,2)$

· first find the parametric equation

$\langle 0,0,0\rangle + t(\langle 2,2,2\rangle - \langle 0,0,0\rangle) = \langle 2t,2t,2t\rangle$

$r'(t) = \langle 2,2,2\rangle \quad \|r'(t)\| = \sqrt{2^2+2^2+2^2} = \sqrt{12}$

$\int_0^2 \frac{8\sqrt{3}(2t)(2t)(2t)}{3}\sqrt{12}\,dt \to \int_0^2 128t^3 \to 32t^4\Big|_0^2 = \boxed{512}$

8.) find $\int_0^2\int_{\sqrt{y/2}}^{1} e^{x^2}\,dx\,dy$

$[(x,y) = 0 \le y \le 2,\ \sqrt{y/2} \le x \le 1]$

$x = \sqrt{y/2} \to x^2 = y/2 \quad y = 2x^2$

$\int_0^1\int_0^{2x^2} e^{x^2}\,dy\,dx \to \int_0^{2x^2} e^{x^2}\,dy \to e^{x^2}y\Big|_0^{2x^2} = 2x^2e^{x^2}$

$\int_0^1 2x^2e^{x^2}\,dx \to u = x^2 \quad 2ue^u \to e^u\Big|_0^1 = \boxed{e-1}$

$2x = dx$

9.) compute $\iiint_E \frac{10(x^2+y^2+z^2)}{9\pi}\,dV$, $E = [(x,y,z) = x^2+y^2+z^2 \le 1]$

$[(\rho,\varphi,\theta) = 0 \le \rho \le 1,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi]$

$x^2+y^2+z^2 = \rho^2$

$\frac{10}{9\pi}\int_0^1\int_0^{\pi}\int_0^{2\pi} \rho^2\rho^2 \sin\varphi\,d\theta\,d\varphi\,d\rho = \frac{10}{9\pi}\int_0^1\int_0^{\pi}\int_0^{2\pi} \rho^2 \sin\varphi\,d\theta\,d\varphi\,d\rho$

$\frac{10}{9\pi}\left(\frac{\rho^5}{5}\Big|_0^1\right)\cdot\left(-\cos\varphi\Big|_0^{\pi}\right)\cdot 2\pi = \frac{2}{9\pi}(2)(2\pi) = \boxed{\frac{8}{9}}$

10.) find $\nabla \cdot F$ if $F = \langle 2\cos(xy), 2\sin(yz), 2\cos(xz) \rangle$

$$\nabla \cdot F = \frac{\partial}{\partial x}(2\cos(xy)) + \frac{\partial}{\partial y}(2\sin(yz)) + \frac{\partial}{\partial z}(2\cos(xz)) =$$

· using def of div

$$\boxed{-2x\sin(xy) + 2y\cos(yz) - 2z\,\cancel{\cos}\sin(xz)}$$