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1. Find the Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space

   x = 2uv + w , y = 2uw + v , z = 2vw + u ,

   at the point (u, v, w) = (2, 2, 2).

$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} 2v & 2u & 1 \\ 2w & 1 & 2u \\ 1 & 2w & 2v \end{pmatrix} = \begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 1 & 4 & 4 \end{pmatrix}$$

$$= 4(1\cdot 4 - 4\cdot 4) - 4(4\cdot 4 - 1\cdot 4) + 1(4\cdot 4 - 1\cdot 1)$$

$$\boxed{= -81}$$

2. (i) (3 points) Show that F = <6x^{2}yz+yz+cos(x + y + z), 2x^{3}z+xz+cos(x + y + z), 2x^{3}y+xy+cos(x + y + z)> , is a conservative vector field

   (ii) (4 point) Find a function f(x, y, z) such that F = gradf.

   (iii) (3 points) Find the line-integral Int(F.dr)|C where C is the curve r = <2sint, 2cost + 1, 2sint >, 0 ≤ t ≤ π .

(i)
$$\begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6x^2yz+yz+\cos(x+y+z), & 2x^3z+xz+\cos(x+y+z), & 2x^3y+xy+\cos(x+y+z) \end{pmatrix}$$

$$i\cdot\left(\frac{\partial}{\partial y}(2x^3y+xy+\cos(x+y+z)) - \frac{\partial}{\partial z}(2x^3z+xz+\cos(x+y+z))\right)$$

$$-j\cdot\left(\frac{\partial}{\partial x}(2x^3y+xy+\cos(x+y+z)) - \frac{\partial}{\partial z}(6x^2yz+yz+\cos(x+y+z))\right)$$

$$k\cdot\left(\frac{\partial}{\partial x}(2x^3z+xz+\cos(x+y+z)) - \frac{\partial}{\partial y}(6x^2yz+yz+\cos(x+y+z))\right)$$

$$\left.\begin{array}{l} i\cdot(2x^3+x+\sin(x+y+z) - 2x^3+x+\sin(x+y+z)) \\ -j\cdot(6x^2y+y+\sin(x+y+z) - 6x^2y+y+\sin(x+y+z)) \\ k\cdot(6x^2z+z+\sin(x+y+z) - 6x^2z+z+\sin(x+y+z)) \end{array}\right\} = i0+j0+k0 = \langle 0,0,0\rangle$$

(ii)
$$\int 6x^2yz+yz+\cos(x+y+z)\,dx = 2x^3yz+xyz+\sin(x+y+z)+\phi(y,z)$$

$2x^3z + xz + \cos(x+y+z) + \phi_y(y,z) = 2x^3z + xz + \cos(x+y+z)$

$\phi_y(y,z) = 0 \Rightarrow \phi(y,z) = \psi(z)$

$f = 2x^3yz + xyz + \sin(x+y+z) + \psi(z)$

$f_z = 2x^3y + xy + \cos(x+y+z) \Rightarrow 2x^3y + xy + \cos(x+y+z) + \psi'(z) = 2x^3y + xy + \cos(x+y+z)$

$\psi(z) = C \Rightarrow C = 0$

$\Rightarrow f(x,y,z) = 2x^3yz + xyz + \sin(x+y+z)$

(iii) $r(0) = (0, 3, 0)$ $\quad r(\pi) = (0, 0, 0)$

$\int_C F \cdot dr = f(0,0,0) - f(0,3,0) = \left(2(0)^3 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 + \sin(0+0+0) - 2(0)^3 \cdot 3 \cdot 0 + 0 \cdot 3 \cdot 0 + \sin(0+3+0)\right)$

$= -\sin 3$

3. Sketch the region of integration and change the order of integration.

Int(Int(F(x, y)dy|0..2e+1)dx|2..4)

$$\int_0^{2e+1} \int_2^4 F(x,y)\,dx\,dy + \int_{2e+1}^{2e^2+1} \int_{\ln(y-2)}^{4} F(x,y)\,dx\,dy$$

$\{(x,y) \mid 2 \le x \le 4,\ 0 \le y \le 2e^x + 1\}$

floor: $(2,0)$ to $(4,0)$

Left wall: $(2,0)$ to $(2, 2e+1)$

Right wall: $(4,0)$ to $(4, 2e^2+1)$

Roof: $y = e^x + 1$

4. Use Largange multipliers (no credit for other methods) to find the smallest value that 2x + 2y + 2z can be, given that xyz = 2.

$f = 2x + 2y + 2z, \quad g = xyz = 2$

$\nabla f = \langle 2, 2, 2 \rangle$, $\nabla g = \langle yz, xz, xy \rangle$

$\nabla f = \lambda \nabla g \Rightarrow \langle 2, 2, 2 \rangle = \lambda \langle yz, xz, xy \rangle$

$2 = \lambda yz \quad 2 = \lambda xz \quad 2 = \lambda xy \qquad xyz = 2$

$x = \lambda xyz \quad y = \lambda xyz \quad z = \lambda xyz$

$x = \lambda \quad y = \lambda \quad z = \lambda \qquad \lambda^3 = 2$

$x + y + z = 2 + 2 + 2 = 6$

5. Compute the volume integral Int(Int(Int(6xyzdV)))|$_E$ where E is the region in 3D {(x, y, z)| 0 ≤ x ≤ y ≤ z ≤ 1}

$$\int_0^1 \int_0^z \int_0^y 6xyz \, dx \, dy \, dz$$

$$\Rightarrow 6yz \int_0^y x \, dx = 3y^3 z \Rightarrow \int_0^z 3y^3 z \, dy = \frac{3}{4} z^5$$

$$\Rightarrow \int_0^1 \frac{3}{4} z^5 \, dz = \frac{1}{8} z^6 \Big|_0^1 = \frac{1}{8}$$

6. By converting to polar coordinates, compute Int(Int((x^2+y^2)^2/240pi dydx)|-sqrt(4-x^2)..sqrt(4-x^2))|-2..2)

$$\{ (r, \theta) \mid 0 \le r \le 2 , \ 0 \le \theta \le 2\pi \}$$

$$x^2 + y^2 = r^2 \Rightarrow \int_0^{2\pi} \int_0^2 \frac{r^4}{240\pi} r \, dr \, d\theta$$

$$\Rightarrow \frac{1}{240\pi} \int_0^{2\pi} \int_0^2 r^5 \, dr \, d\theta \Rightarrow \int_0^{2\pi} \frac{r^6}{6} \, d\theta \Big|_0^2 = \frac{1}{240\pi} (2\pi) \left(\frac{64}{6} \right)$$

$$= \frac{128}{1440} = \frac{4}{45}$$

7. Compute the line integral Int(3√(2)xyz/2ds)|$_C$, where C is the line-segment joining (0, 0, 0) and (1, 1, 1)

$$\int_C \frac{3\sqrt{3}\;xyz}{2}\,ds$$

$x = \langle 0,0,0\rangle + t\langle 1,1,1\rangle - \langle 0,0,0\rangle = \langle t,t,t\rangle,\quad 0 \le t \le 1$

$r'(t) = \langle 1,1,1\rangle \Rightarrow \|r'(t)\| = \sqrt{3}$

$$\int_0^1 \frac{3\sqrt{3}\;t^3}{2}\sqrt{3}\,dt = \int_0^1 \frac{3}{2}t^3\,dt = \frac{3}{8}t^4\Big|_0^1 = \frac{3}{8}$$

8. Compute

Int(Int(e^x^2)|sqrt(y/2)..1)|0..2)

$\{(x,y) \mid 0 \le y \le 2,\ y/2 \le x \le 1\}$ $y = 2x$

$\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2x\}$

$$\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx \Rightarrow \int_0^{2x} e^{x^2}\,dy = 2x\,e^{x^2}$$

$$\Rightarrow \int_0^1 2x\,e^{x^2}\,dx = \int_0^1 e^u\,du = e-1$$

9. Compute the volume integral

Int(Int(Int$_E$(2(x^2+y^2+z^2)/pi dV)))

Where

E = {(x, y, z)| $x^2 + y^2 + z^2 \le 1$}

10. Find divF if F = <sin(2xy), sin(2yz), sin(2xz)>

$$\nabla \cdot F = \frac{\partial}{\partial x}(\sin 2xy) + \frac{\partial}{\partial y}(\sin 2yz) + \frac{\partial}{\partial z}(\sin 2yz)$$

$$2y\cos 2xy + 2z\cos 2yz + 2y\cos 2yz$$

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