1. (10 pts.)
Find the Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space.

$$x = vw + u \,, \quad y = uw \quad , \quad z = uv + u \quad ,$$

at the point $(u, v, w) = (1, 3, 2)$.

The **type** of the answers is: expression

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ans. $u^2 + uw + 2uvw$

$$\frac{\partial x}{\partial u}\begin{vmatrix} \frac{\partial y}{v} & \frac{\partial y}{w} \\ \frac{\partial z}{v} & \frac{\partial z}{w} \end{vmatrix} - \frac{\partial x}{\partial v}\begin{vmatrix} \frac{\partial y}{u} & \frac{\partial y}{w} \\ \frac{\partial z}{u} & \frac{\partial z}{w} \end{vmatrix} + \frac{\partial x}{\partial w}\begin{vmatrix} \frac{\partial y}{u} & \frac{\partial y}{v} \\ \frac{\partial z}{u} & \frac{\partial z}{v} \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & u \\ u & 0 \end{vmatrix} - w\begin{vmatrix} w & u \\ v+1 & 0 \end{vmatrix} + v\begin{vmatrix} w & 0 \\ v+1 & u \end{vmatrix}$$

$$= 1\,(u^2 - 0) - w\,(0 - u(v+1)) + v\,(wu - 0)$$

$$= u^2 + uvw + uw + uvw = \boxed{u^2 + uw + 2uvw}$$

2. (10 points altogether)
(i) (3 points) Show that

$\mathbf{F} = \langle\ 4xy + xz - \sin(x+z)\ ,\ x^4y + yz - \sin(x+z)\ ,\ x^4yz + x - \sin(x+z)\ \rangle$,

is a conservative vector field.

(ii) (4 point) Find a function $f(x, y, z)$ such that $\mathbf{F} = \bigtriangledown f$.

(iii) (3 points) Find the line-integral $\int_C \mathbf{F}.d\mathbf{r}$ where C is the curve

$$\mathbf{r} = \langle\ \sin t,\ \sin t + 1\ ,\ \cos 2t \rangle \quad , \quad 0 \leq t \leq \pi \quad .$$

The **types** of the answer is: For (ii) function For (iii) number

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answers (ii) f(x,y,z)= DNE (function not conservative)

(iii) 23

(i) $\frac{\partial}{\partial y} F_x = 4x \neq \frac{\partial}{\partial x} F_y = 4xy + \cos(x+z)$ not conservative

(ii) $f = \int 4xy + xz - \sin(x+z)\ dx = 2x^2y + \frac{x^2z}{2} + \cos(x+z) + g(y,z)$

$\frac{\partial}{\partial y}(2x^2y + \frac{x^2z}{2} + \cos(x+z) + g(y,z))$

$2x^2 + g_y = x^4y + yz - \sin(x+z)$

$g_y = x^4y + yz - \sin(x+z) - 2x^2$

DNE

$g(y,z) = \int g_y\ dy = \frac{x^4y^2}{2} + \frac{y^2z}{2} - y\sin(x+z) - 2x^2y$

$f = 2x^2y + \frac{x^2z}{2} + \cos(x+z) + \frac{x^4y^2}{2} + \frac{y^2z}{2} - y\sin(x+z) - 2x^2y$

(iii) $\int_0^\pi \mathbf{F} \cdot d\mathbf{r}$ $\quad d\mathbf{r} = \langle -\cos t, -\cos t, \frac{1}{2}\sin 2t \rangle$

$\int_0^\pi \langle 4xy + xz - \sin(x+z),\ x^4y + yz - \sin(x+z),\ x^4yz + x - \sin(x+z) \rangle \cdot \langle -\cos t, -\cos t, \frac{1}{2}\sin 2t \rangle$

$\int_0^\pi -\cos t\,(4xy + xz - \sin(x+z)) - \cos t\,(x^4y + yz - \sin(x+z)) + \frac{1}{2}\sin 2t\,(x^4yz + x - \sin(x+z)) \rangle$

$= 23$

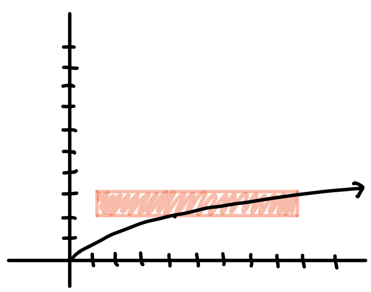
3. (10 points)
Sketch the region of integration and change the order of integration.

$$\int_1^9 \int_2^{\sqrt{x}} F(x, y)\, dy\, dx$$

The **type** of the answer is: Range

ans. $D: \{(x, y) \mid 2 \le y \le 3,\ \frac{y}{2} \le x \le 9\}$

$D: \{(x, y) \mid 1 \le x \le 9,\ 2 \le y \le \sqrt{x}\}$



$D: \{(x, y) \mid 2 \le y \le 3,\ \frac{y}{2} \le x \le 9\}$

4. (10 points) Use Largange multipliers (no credit for other methods) to find the smallest value that $2xyz$ can be, given that $2x+y+z=4$

The **type** of the answer is: number

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ans. $\frac{64}{27}$

$\nabla f = \langle 2yz, 2xz, 2xy \rangle$

$\nabla g = \langle 2, 1, 1 \rangle$

$\langle 2yz, 2xz, 2xy \rangle = \lambda \langle 2, 1, 1 \rangle$

$2yz = \lambda 2$ $\quad$ $2xz = \lambda$ $\quad$ $2xy = \lambda$

$\lambda = yz$ $\quad$ $\lambda = 2xz$ $\quad$ $\lambda = 2xy$

$yz = 2xz$ $\quad$ $2xz = 2xy$

$y = 2x$ $\quad$ $y = z$

$z = 2x$

$2x + 2x + 2x = 4$

$6x = 4$

$x = \frac{2}{3} \rightarrow y = \frac{4}{3} \rightarrow z = \frac{4}{3}$

$f\left(\frac{2}{3}, \frac{4}{3}, \frac{4}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = \frac{64}{27}$

5. (10 points) Compute the volume integral

$$\int\int\int_E yz\sin(x)\, dV$$

where E is the region in 3D

$$\{(x, y, z)\,|\, 0 \le x \le y \le z \le 1\} \quad .$$

The **type** of the answer is: number

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ans. 0.015

$$\underbrace{\int_0^1\int_0^1\int_0^1 yz\sin(x)}\ dz\,dy\,dx$$

$$\frac{yz^2\sin(x)}{2}\Big|_0^1 = \frac{y\sin(x)}{2}$$

$$\int_0^1 \frac{y\sin(x)}{2}\ dy$$

$$= \frac{y^2\sin(x)}{4}\Big|_0^1 = \frac{\sin(x)}{4}$$

$$\frac{1}{4}\int_0^1 \sin(x)$$

$$= \frac{1}{4}\left(-\cos(x)\Big|_0^1\right) = \frac{1}{4}(-0.54+1) \approx \frac{0.06}{4} \approx 0.015$$

6. (10 points) By converting to polar coordinates, compute

$$\int_0^1 \int_0^1 9(x^2+y^2)\, dy\, dx$$

The **type** of the answer is: number

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ans. $\frac{9\pi}{8}$

$9(x^2+y^2)$
$9r^2$

$D: \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1 \}$

$D: \{ (r,\theta) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1 \}$

$$\int_0^{\frac{\pi}{2}} \int_0^1 9r^2 \; r\,dr\,d\theta$$

$$\int_0^1 9r^3\,dr = \frac{9}{4}r^4 \Big|_0^1 = \frac{9}{4}$$

$$\int_0^{\frac{\pi}{2}} \frac{9}{4}\,d\theta = \frac{9}{4}\theta \Big|_0^{\frac{\pi}{2}} = \frac{9\pi}{8}$$

7. (10 points) Compute the line integral

$$\int_C 4xyz \, ds \quad ,$$

where C is the line-segment joining $(4, 2, 1)$ and $(0, 0, 3)$

The **type** of the answer is: number

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ans. 113.49

$PQ = (4,2,1) + t(-4,-2,2)$

$= \langle 4-4t, 2-2t, 1+2t \rangle$

$x = 4-4t \quad y = 2-2t \quad z = 1+2t$

$dx = -4 \quad dy = -2 \quad dz = 2$

$ds = \sqrt{16+4+4} = \sqrt{24}\, dt$

$\int_0^1 4(4-4t)(2-2t)(1+2t)\sqrt{24}\, dt$

$\sqrt{24}\int_0^1 (16-16t)(2-2t)(1+2t)\, dt = \sqrt{24}\int_0^1 (42-42t-32t+32t^2)(1+2t)\, dt$

$= \sqrt{24}\int_0^1 32t^2 - 75t + 42 \; (1+2t)\, dt = \sqrt{24}\int 32t^2 + 64t^3 - 75t - 150t^2 + 42 + 84t \; dt$

$= \sqrt{24}\int_0^1 64t^3 - 118t^2 + 9t + 42 \; dt = \sqrt{24}\left(16t^4 - \frac{118}{3}t^3 + \frac{9}{2}t^2 + 42t \Big|_0^1\right)$

$= \sqrt{24}\left(16 - \frac{118}{3} + \frac{9}{2} + 42\right) = 113.49$

8. (10 points) Compute

$$\int_0^3 \int_{y^3}^1 e^x \quad dx\,dy \quad .$$

(Hint: Not even Dr. Z. can do $\int e^{x^3}\,dx$, so you must be clever, and first change the order of integration.)

The **type** of the answer is: number

ans. $\frac{3}{2}e$

$D = \{(x,y) \mid 0 \le y \le 3 \,,\ y^3 \le x \le 1\}$

$D = \{(x,y) \mid 0 \le x \le 1 \,,\ 0 \le y \le 3x\}$

$$\int_0^1 \int_0^{3x} e^x \ dy\,dx \quad = \quad ye^x \Big|_0^{3x} = \ 3xe^x$$

$$\int_0^1 3xe^x \ dx \quad = \quad \frac{3}{2}x^2e^x \Big|_0^1 = \ \frac{3}{2}e$$

9. (10 points) Compute the volume integral

$$\iiint_E x^2 + y^2 + z \quad dV \quad ,$$

where

$$E = \{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 4\} \quad .$$

The **type** of the answer(s) is: number / value

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ans. 4ln4 - 3

$$x = \sqrt{4 - y^2 - z^2} \qquad y = \sqrt{4 - z^2} \qquad z = \sqrt{4} = 2$$

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} x^2 + y^2 + z \quad dx\, dy\, dz$$

$$\frac{x^3}{3} + xy^2 + xz \Big|_0^{\sqrt{4-y^2-z^2}} = \frac{(4-y^2-z^2)\sqrt{4-y^2-z^2}}{3} + \frac{3\sqrt{4-y^2-z^2}\,(y^2 - z)}{3}$$

$$= \frac{1}{3}\left(\sqrt{4-y^2-z^2}\,(4 \underbrace{- y^2 - z^2 + 3y^2}_{2y^2} - 3z)\right)$$

$$\frac{1}{3}\int_0^4 (4-y^2-z^2)^{1/2}\,(2y^2 - z^2 - 3z + 4)\, dy$$

$$= 4\ln 4 - 3$$

10. (10 points) Find $\nabla \cdot \mathbf{F}$ if

$$\mathbf{F} = \langle \sin(xy)\,, \sin(yz)\,, \sin(xz)\,\rangle \quad .$$

The **type** of the answer is: vector

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ans. $\langle \cos(xy), \cos(yz), \cos(xz) \rangle$

$\nabla F = \langle \frac{\partial}{\partial x} \sin(xy), \frac{\partial}{\partial y} \sin(yz), \frac{\partial}{\partial z} \sin(xz) \rangle$

$\nabla F = \langle \cos(xy), \cos(yz), \cos(x,z) \rangle$