Lecture #21 Quiz

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Section: ____ **E-Mail address:** ________________

MATH 251 (4,6,7), Dr. Z. , Exam 2, Tue., Nov. 21, 2017, SEC 118

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM
Do not write below this line

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1. (out of 10)

2. (out of 10)

3. (out of 10)

4. (out of 10)

5. (out of 10)

6. (out of 10)

7. (out of 10)

8. (out of 10)

9. (out of 10)

10. (out of 10)

MAKE SURE TO PUT THE TYPE!
Types: Number, Function of *variable*(s), 2D vector of numbers, 3D vector of numbers, 2D vector of functions (aka 2D vector-field), 3D vector of functions (aka 3D vector field), equation of a plane, parametric equation of a line, equation of a line, equation of a surface, equation of a line, DNE (does not exist), abstract double-integral, abstract triple-integral.

1. (10 pts.)
Find the Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space.

$$x = uv + w \quad , \quad y = uw + v \quad , \quad z = vw + u \quad ,$$

at the point $(u, v, w) = (2, 2, 2)$.

The **type** of the answers is: number

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ans.

-5

$$\begin{vmatrix} v & u & 1 \\ w & 1 & u \\ 1 & w & v \end{vmatrix} = v(v - wu) - u(wv - u) + (w^2 - 1)$$

Plug in: $(2,2,2)$

$$J = 2(2-4) - 2(4-2) + (4-1) = -4 - 4 + 3 = \boxed{-5}$$

Made Up:

$x = 2u + w \qquad y = uw + v^2 \qquad w = v + u \qquad$ at $(1,2,3)$

$$\begin{vmatrix} 2 & 0 & 1 \\ w & 2v & u \\ 1 & 1 & 0 \end{vmatrix} = 2(0 - u) - 0 + (w - 2v)$$

$$= 2(-u) + w - 2v$$

$$J_{(1,2,3)} = -2 + 3 - 2(2) = \boxed{-3}$$

2. (10 points altogether)
(i) (3 points) Show that

$$\mathbf{F} = \langle\, 3\,x^2yz{+}yz{+}\cos\,(x+y+z)\ ,\ x^3z{+}xz{+}\cos\,(x+y+z)\ ,\ x^3y{+}xy{+}\cos\,(x+y+z)\,\rangle \quad ,$$

is a conservative vector field.

(ii) (4 point) Find a function $f(x,y,z)$ such that $\mathbf{F} = \bigtriangledown f$.

(iii) (3 points) Find the line-integral $\int_C \mathbf{F}.d\mathbf{r}$ where C is the curve

$$\mathbf{r} = \langle \sin t, \cos t + 1, \sin 2t \rangle \quad , \quad 0 \le t \le \pi \quad .$$

The **types** of the answer is: For (ii) For (iii)

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answers (ii) f(x,y,z)=

(iii)

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i)

$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= 3x^2z + z - \sin(x+y+z) \\ \frac{\partial F_2}{\partial x} &= 3x^2z + x - \sin(x+y+z) \end{aligned}\right\} =$$

$$\left.\begin{aligned} \frac{\partial F_2}{\partial z} &= x^3 + x - \sin(x+y+z) \\ \frac{\partial F_3}{\partial F y} &= x^3 + x - \sin(x+y+z) \end{aligned}\right\} =$$

$\therefore$ conservative ✓

$$\left.\begin{aligned} \frac{\partial F_3}{\partial x} &= 3x^2y + y - \sin(x+y+z) \\ \frac{\partial F_1}{\partial z} &= 3x^2y + y - \sin(x+y+z) \end{aligned}\right\} =$$

ii)

$$f(x,y,z) = \int 3x^2yz + yz + \cos(x+y+z)\, dx = \cancel{3}yz\frac{x^3}{\cancel{3}} + yzx + \sin(x+y+z) + g(y,z)$$

$$\frac{\partial}{\partial y}\left(yzx^3 + yzx + \sin(x+y+z) + g(y,z)\right) = zx^3 + zx + \cos(x+y+z) + g_y(y,z)$$

$$zx^3 + zx + \cos(x+y+z) + g_y(y,z) = x^3z + xz + \cos(x+y+z)$$

$$g_y(y,z) = 0 \quad \therefore \quad g(y,z) = C + h(z)$$

$$\frac{\partial}{\partial z}\left(yzx^3 + yzx + \sin(x+y+z) + C + h(z)\right) = x^3y + zx + \cos(x+y+z) + h'(z)$$

$$x^3y + zx + \cos(x+y+z) + h'(z) = x^3y + zx + \cos(x+y+z)$$

$$\therefore h'(z) = 0 \longrightarrow h(z) = C$$

$$\boxed{f(x,y,z) = yzx^3 + yzx + \sin(x+y+z)}$$

ii) $\int_C F \cdot dr = f(r(\pi)) - f(r(0))$ $\quad r(\pi): 0, 0, 0$

$\quad r(0): 0, 2, 0$

$= 0 - \sin(2)$

$= -\sin(2)$

Made Up:

$$F = \langle 2, x, y^3z^4, \; 3x^2y^2z^4, \; 4x^2y^3z^3 \rangle$$

i) $\frac{\partial F_1}{\partial y} = 6y^2xz^4$

$\frac{\partial F_2}{\partial x} = 6xy^2z^4$

$\frac{\partial F_2}{\partial z} = 12z^3x^2y^2$

$\frac{\partial F_3}{\partial Fy} = 12x^2y^2z^3$

$\frac{\partial F_3}{\partial x} = 8xy^3z^3$

$\frac{\partial F_1}{\partial z} = 8xy^3z^3$

$\therefore$ conservative

ii)

$$f(x,y,z) = \int 2xy^3z^4 \, dx = x^2y^3z^4 + g(y,z)$$

$$\frac{\partial}{\partial y}\left(x^2y^3z^4 + g(y,z)\right) = 3x^2y^2z^4 + g_y(y,z) = 3x^2y^2z^4$$

$$\therefore \; g_y = 0 \longrightarrow g(y,z) = c + h(z)$$

$$\frac{\partial}{\partial z}\left(x^2y^3z^4 + h(z)\right) = 4z^3x^2y^3 + h'(z) = 4z^3x^2y^3$$

$$\therefore \; h'(z) = 0 \qquad h(z) = c$$

$$f(x,y,z) = x^2y^3z^4$$

at $c=0$

iii) $r(t) = \langle 2t+1, \; t^2+t, \; 3t+t^2 \rangle \qquad 0 \le t \le 1$

$$r(0) = \langle 1, 0, 0 \rangle \qquad r(1) = \langle 3, 2, 4 \rangle$$

$$\cancel{F(r(t)) = \langle 2(2t+1)(t^2+t)^3(3t+t^2)^4, \; 3(2t+1)^2(t^2+t)^2(3t+t^2)^4, \; 4(2t+1)^2(t^2+t)^3(3t+t^2)^3 \rangle}$$

$$\int F \cdot dr = F(r(1)) - F(r(0)) = (2(3)(2)^3(4)^4 + 3(3)^2(2)^2(4)^4 + 4(3)^2(2)^3(4)^3 - 0$$

$$= 12{,}288 + 27{,}648 + 18{,}432 = \boxed{58{,}368}$$

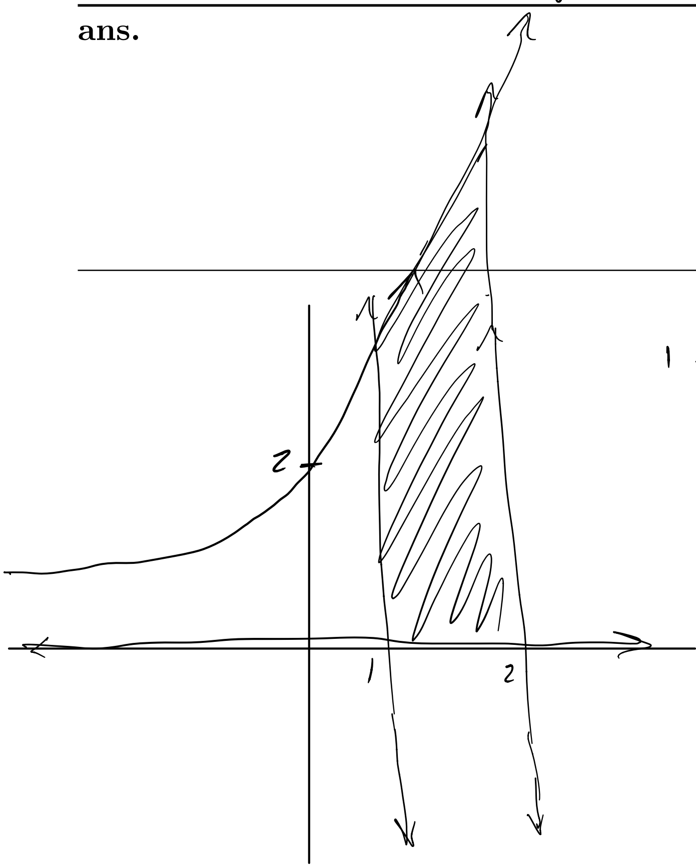
$$F = \langle 2xy^3z^4, \; 3x^2y^2z^4, \; 4x^2y^3z^3 \rangle$$

3. (10 points)
Sketch the region of integration and change the order of integration.

$$\int_1^2 \int_0^{e^x+1} F(x,y)\, dy\, dx$$

The **type** of the answer is: integral

ans.



$1 \leq x \leq 2 \qquad 0 \leq y \leq e+1$

$$\int_0^{e+1} \int_1^2 F(x,y)\, dx\, dy$$

$x = 1$ to 2

4. (10 points) Use Largange multipliers (no credit for other methods) to find the smallest value that $x + y + z$ can be, given that $xyz = 1$.

The **type** of the answer is:

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ans.

$\nabla f = \langle 1,1,1 \rangle \qquad g\nabla = \langle yz, xz, xy \rangle$

$\langle 1,1,1 \rangle = \lambda \langle yz, xz, xy \rangle$

$\langle 1,1,1 \rangle = \langle yz\lambda + xz\lambda + xy\lambda \rangle$

$1 = yz\lambda \qquad 1 = xz\lambda \qquad 1 = xy\lambda$

$\lambda = \frac{1}{yz} \qquad \lambda = \frac{1}{xz} \qquad \lambda = \frac{1}{xy}$

$\frac{1}{yz} = \frac{1}{xz} = \frac{1}{xy}$

$xz = yz \qquad xy = xz$

$x = y \qquad y = z$

$x = y = z$

$x^3 = 1$

$x = 1$

since $x = y = z$ and $x = 1$, $z = 1$ and $y = 1$

$f(1,1,1) = 1 + 1 + 1 = \boxed{3}$

Made up: $f = 3x + 2y + 4z$, $x^2 + 2y^2 + 6z^2 = 1$

$\nabla f = \langle 3, 2, 4 \rangle$ $\quad \nabla g = \langle 2x, 4y, 12z \rangle$

$\langle 3, 2, 4 \rangle = \lambda \langle x, 2y, 6z \rangle$

$3 = \lambda x \quad 2 = \lambda 2y \quad 4 = \lambda 6z$

$\lambda = 3/x \quad \lambda = 1/y \quad \lambda = 2/3z$

$\frac{3}{x} = \frac{1}{y} = \frac{2}{3z}$

$3y = x \qquad 3z = 2y$

$9z = 2x \qquad 2y = 3z$

$x = \frac{9z}{2} \qquad y = \frac{3z}{2}$

$\left(\frac{9z}{2}\right)^2 + 2\left(\frac{3z}{2}\right)^2 + 6z^2 = 1$

$\frac{81z^2}{4} + \frac{18z^2}{4} + 36z^2 = 1$

$z = \pm \frac{2}{\sqrt{123}}$

$x = \frac{9}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{123}} = \frac{9}{\sqrt{123}} \qquad y = \frac{3}{2} \cdot \frac{2}{\sqrt{123}} = \frac{3}{\sqrt{123}}$

$x = \frac{9}{\cancel{2}} \cdot \left(-\frac{\cancel{2}}{\sqrt{123}}\right) = -\frac{9}{\sqrt{123}} \qquad y = \frac{3}{2}\left(-\frac{2}{\sqrt{123}}\right) = -\frac{3}{\sqrt{123}}$

$f_1 = 3\left(\frac{9}{\sqrt{123}}\right) + 2\left(\frac{3}{\sqrt{123}}\right) + 4\left(\frac{2}{\sqrt{123}}\right)$

$f_2 = 3\left(-\frac{9}{\sqrt{123}}\right) - 2\left(\frac{3}{\sqrt{123}}\right) - 4\left(\frac{2}{\sqrt{123}}\right)$

5. (10 points) Compute the volume integral

$$\int\int\int_E 48\,x\,y\,z\,dV$$

where E is the region in 3D

$$\{(x, y, z)\,|\,0 \le x \le y \le z \le 1\} \quad .$$

The **type** of the answer is:

ans.

$$\int_0^1 \int_0^z \int_0^y 48xyz \; dx\,dy\,dz$$

$$\int_0^y 48xyz\,dx = 24x^2yz \Big|_0^y = 24y^2yz = 24zy^3$$

$$\int_0^z 24y^3z\,dy = 6y^4z \Big|_0^z = 6z^4z = 6z^5$$

$$\int_0^1 6z^5\,dz = z^6 \Big|_0^1 = \boxed{1}$$

6. (10 points) By converting to polar coordinates, compute

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{\left(x^2+y^2\right)^2}{243\pi}\,dy\,dx$$

The **type** of the answer is:

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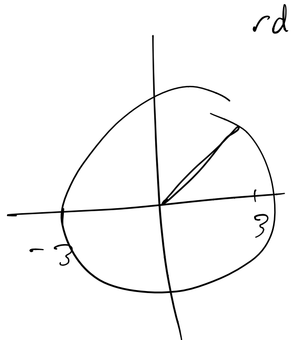
ans.

$r\,dr\,d\theta$



full circle

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^3 \frac{(r^2)^2}{243\pi}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \frac{r^5}{243\pi}\,dr\,d\theta$$

$$\int_0^3 \frac{r^5}{243\pi}\,dr = \frac{1}{243\pi}\cdot \frac{r^6}{6}\Big|_0^3 = \frac{1}{\cancel{243}\pi}\cdot\frac{\cancel{243}}{2}$$

$$= \frac{1}{2\pi}$$

$$\int_0^{2\pi} \frac{1}{2\pi}\,d\theta = \frac{\theta}{2\pi}\Big|_0^{2\pi} = \boxed{1}$$

7. (10 points) Compute the line integral

$$\int_C \frac{4\sqrt{3}\,x\,y\,z}{3}\,ds \quad ,$$

where C is the line-segment joining $(0,0,0)$ and $(1,1,1)$

The **type** of the answer is:

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ans.

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parametrize: $(1-t)(0,0,0) + t(1,1,1)$

$(0,0,0) + (t,t,t)$

$r(t) = \langle t, t, t \rangle$

$r'(t) = \langle 1,1,1 \rangle$

$\|r'(t)\| = \sqrt{1+1+1} = \sqrt{3}$

$f = \frac{4\sqrt{3}\,xyz}{3}$ $\quad f(r(t)) = \frac{4\sqrt{3}\,t^3}{3} \cdot \sqrt{3} = 4t^3$

$\int_0^1 4t^3\,dt = t^4 \Big|_0^1 = \boxed{1}$

8. (10 points) Compute

$$\int_0^3 \int_{\sqrt{y/3}}^1 e^{x^3}\, dx\, dy \quad .$$

(Hint: Not even Dr. Z. can do $\int e^{x^3}\, dx$, so you must be clever, and first change the order of integration.)

The **type** of the answer is:

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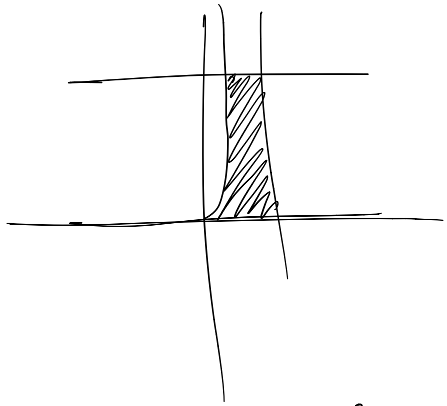
ans.



$0 \le y \le 3$ $\qquad$ $0 \le x \le 3x^2$

$x = \sqrt{\frac{y}{3}}$

$x^2 = \frac{y}{3}$

$3x^2 = y$

$$\int_0^1 \int_0^{3x^2} e^{x^3}\, dy\, dx$$

$$\int_0^{3x^2} e^{x^3}\, dy = e^{x^3} y \Big|_0^{3x^2} = 3x^2 e^{x^3}$$

$$\int_0^1 3x^2 e^{x^3}\, dx = e^{x^3} \Big|_0^1 = \boxed{e-1}$$

$u = x^3 \quad du = 3x^2 dx$

9. (10 points) Compute the volume integral

$$\int\int\int_E \frac{5\,(x^2+y^2+z^2)}{4\pi}\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,x^2\,+\,y^2\,+\,z^2 \leq 1\} \quad .$$

The **type** of the answer(s) is:

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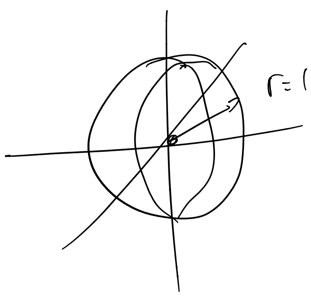
ans.



$0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \frac{5(\rho^2)}{4\pi}\,\rho^2 \sin\phi \,d\rho\,d\phi\,d\theta$$

$$\int_0^{1} \frac{5(\rho^2)}{4\pi}\,\rho^2 \sin\phi\,d\rho = \frac{\cancel{5}\sin\phi}{4\pi}\cdot\frac{\rho^5}{\cancel{5}}\Big|_0^1 = \frac{\sin\phi}{4\pi}$$

$$\int_0^{\pi} \frac{\sin\phi}{4\pi}\,d\phi = -\frac{1}{4\pi}\cos\phi\Big|_0^{\pi} = -\frac{(-1)}{4\pi} + \frac{1}{4\pi} = \frac{1}{2\pi}$$

$$\int_0^{2\pi} \frac{1}{2\pi}\,d\theta = \frac{\theta}{2\pi}\Big|_0^{2\pi} = \boxed{1}$$

10. (10 points) Find $\bigtriangledown \cdot \mathbf{F}$ if

$$\mathbf{F} = \langle \sin(xy)\,, \sin(yz)\,, \sin(xz) \rangle \quad .$$

The **type** of the answer is:

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ans.

$$\nabla \cdot F = \langle \cos(xy) \cdot y \,,\ \cos(yz) \cdot z \,,\ \cos(xz) \cdot x \rangle$$

$$= \langle y\cos(xy) \,, z\cos(yz) \,,\ x\cos(x,z) \rangle$$

Made Up: $F = \langle x^2 y \,,\ y^2 z \,,\ z^2 x \rangle$

$$\nabla f = \langle 2xy \,,\ 2yz \,,\ 2zx \rangle$$