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Section 24

Attendance Quiz 21 - 2017 Sample Exam

1) Find the Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space.

$x = u^2 v \quad y = v^2 w \quad z = w^2 u$

at the point $(u, v, w) = (3, 2, 1)$

Type of answer: Number

Answer: 324

The formula for the Jacobian of the transformation is:

$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

So first, find all of the partial derivatives needed:

$x_u = 2uv \quad x_v = u^2 \quad x_w = 0$

$y_u = 0 \quad y_v = 2vw \quad y_w = v^2$

$z_u = w^2 \quad z_v = 0 \quad z_w = 2wu$

Find their numerical values at point $(3, 2, 1)$:

$x_u(3,2,1) = 12 \quad x_v(3,2,1) = 9 \quad x_w(3,2,1) = 0$

$y_u(3,2,1) = 0 \quad y_v(3,2,1) = 4 \quad y_w(3,2,1) = 4$

$z_u(3,2,1) = 1 \quad z_v(3,2,1) = 0 \quad z_w(3,2,1) = 6$

Plug the results into the Jacobian matrix:

$$J(3,2,1) = \begin{vmatrix} 12 & 9 & 0 \\ 0 & 4 & 4 \\ 1 & 0 & 6 \end{vmatrix} = 12(24-0) - 9(0-4) + 0(0-4) =$$

$$= 288 + 36 = \boxed{324}$$

2) (i) Show that:

$$F = \langle e^x, e^y, e^z \rangle$$

is a conservative vector field

(ii) Find a function $f(x,y,z)$ such that $F = \nabla f$

(iii) Find the line integral $\int_C F \cdot dr$ where C is the curve

$$\vec{r} = \langle t^2, 2t, 3t^3 \rangle,\ 0 \le t \le 1$$

<u>Types of answers</u>: (ii) Function of x, y, and z (iii) Number

<u>Answers</u>: (ii) $f(x,y,z) = e^x + e^y + e^z$ (iii) $e + e^2 + e^3 - 3$

(i) If we have a vector field $F = \langle P, Q, R \rangle$, it is conservative if $R_y = Q_z$, $P_z = R_x$ and $Q_x = P_y$.

So, check all of the partial derivatives:

$R_y = 0 = Q_z$, $P_z = 0 = R_x$, $Q_x = 0 = P_y$ ✓

So, F is conservative

(ii) Because $F = \nabla f$, it means that $F = \langle f_x, f_y, f_z \rangle$. So, we can start by integrating $F[1]$ with respect to x:

$\int F[1]\,dx = \int e^x dx = e^x + g(y,z)$, where $g(y,z)$ is like an arbitrary constant.

We can differentiate the result with respect to y and set it equal to $F[2]$ to find $g(y,z)$:

$g_y = e^y \rightarrow g(y,z) = e^y \rightarrow f(x,y,z) = e^x + e^y + h(z)$

We can now differentiate the result with respect to z and set it equal to $F[3]$ to find $h(z)$:

$h_z = e^z \rightarrow h(z) = e^z \rightarrow \boxed{f(x,y,z) = e^x + e^y + e^z}$

(iii) Because F is conservative, it is path independent. Meaning that, no matter the path taken from point A to B, the answer is the same:

$$\int_C F \cdot dr = f(b) - f(a)$$

So first, we need to find the start and end points:

$r(0) = \langle 0^2, 2(0), 3(0)^3 \rangle = \langle 0,0,0 \rangle$

$r(1) = \langle 1^2, 2(1), 3(1)^3 \rangle = \langle 1,2,3 \rangle$

$f(1,2,3) - f(0,0,0) = e + e^2 + e^3 - e^0 - e^0 - e^0 =$

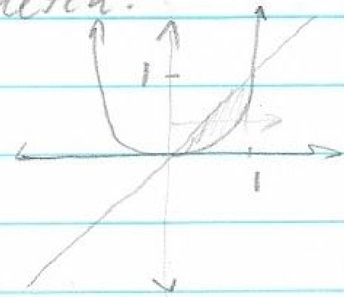
$= \boxed{e + e^2 + e^3 - 3}$

3) Sketch the region and change the order of integration $\int_0^1 \int_{x^2}^{x} F(x,y)\,dy\,dx$

<u>Type of answer</u>: Abstract double-integral

<u>Answer</u>: $\int_0^1 \int_y^{\sqrt{y}} F(x,y)\,dx\,dy$

Sketch:



When we draw a line from left to right of the region, it reaches $x=y$ first, and then $y=x^2$ (or $x=\sqrt{y}$).

So, the inner integral's bounds would be from y to $\sqrt{y}$.

The graphs intersect at $y=0$ and $y=1$, so, the outer integral's bounds would be from 0 to 1.

Our new integral is:

$$\boxed{\int_0^1 \int_y^{\sqrt{y}} F(x,y)\,dx\,dy}$$

4) Use Lagrange multipliers (no credit for other methods) to find the smallest value that $3x-5y+4z$ can be, given that $x^2+y^2+z^2=9$

Type of answer: Number

Answer: $-15\sqrt{2}$

Using a constant variable λ:

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

Using our functions:

$\nabla f(x,y,z) = \langle 3,-5,4 \rangle$

$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$

$3 = 2\lambda x \;,\; -5 = 2\lambda y ,\; 4 = 2\lambda z$

Find x^2, y^2, and z^2 to plug into the constraint:

$2\lambda x = 3 \rightarrow x = \frac{3}{2\lambda} \rightarrow x^2 = \left(\frac{3}{2\lambda}\right)^2 = \frac{9}{4\lambda^2}$

$2\lambda y = -5 \rightarrow y = \frac{-5}{2\lambda} \rightarrow y^2 = \left(\frac{-5}{2\lambda}\right)^2 = \frac{25}{4\lambda^2}$

$2\lambda z = 4 \rightarrow z = \frac{4}{2\lambda} \rightarrow z^2 = \left(\frac{4}{2\lambda}\right)^2 = \frac{16}{4\lambda^2}$

Plug x^2, y^2, and z^2 into the constraint:

$x^2+y^2+z^2 = \frac{9}{4\lambda^2} + \frac{25}{4\lambda^2} + \frac{16}{4\lambda^2} = \frac{50}{4\lambda^2} = 9 \rightarrow$

$\rightarrow \frac{50}{9} = 4\lambda^2 \rightarrow \lambda^2 = \frac{25}{18} \rightarrow \lambda = \pm \frac{5}{3\sqrt{2}}$

Plug the resulting Lagrange multiplier into equations for x, y, and z:

If $\lambda = \frac{5}{3\sqrt{2}}$: $x = \frac{9}{5\sqrt{2}}$, $y = -\frac{3}{\sqrt{2}}$, $z = \frac{6\sqrt{2}}{5} \rightarrow \left(\frac{9}{5\sqrt{2}}, -\frac{3}{\sqrt{2}}, \frac{6\sqrt{2}}{5}\right)$

If $\lambda = -\frac{5}{3\sqrt{2}}$: $x = \frac{-9}{5\sqrt{2}}$, $y = \frac{3}{\sqrt{2}}$, $z = \frac{-6\sqrt{2}}{5} \rightarrow \left(\frac{-9}{5\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{-6\sqrt{2}}{5}\right)$

Plug each point into $f(x,y,z)$ and pick the smallest value:

$f\left(\frac{9}{5\sqrt{2}}, -\frac{3}{\sqrt{2}}, \frac{6\sqrt{2}}{5}\right) = \frac{27}{5\sqrt{2}} + \frac{15}{\sqrt{2}} + \frac{24\sqrt{2}}{5} = 15\sqrt{2}$

$f\left(-\frac{9}{5\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{-6\sqrt{2}}{5}\right) = -\frac{27}{5\sqrt{2}} - \frac{15}{\sqrt{2}} - \frac{24\sqrt{2}}{5} = \boxed{-15\sqrt{2}}$

5) Compute the volume integral:

$$\iiint_E 3x^2y^2z^2\,dV$$

Where E is the region in 3D:

$$\{(x,y,z) \mid x^2+y^2 \le z \le 1,\ y \ge 0,\ x \ge 0\}$$

Type of answer: Number

Answer: $\frac{\pi}{192}$

First, we need the projection of $z = x^2+y^2$ onto the xy-plane to find x and y bounds:

$x^2+y^2=1 \rightarrow$ Equation for a disc of radius 1, centered at the origin.

So, the integral is written as:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 3x^2y^2z^2\,dz\,dy\,dx$$

Solve the inner integral using Maple:

int$(3 \cdot x^2 \cdot y^2 \cdot z^2, z=(x^2+y^2)..1)$; $\rightarrow x^2y^2(1-(x^2+y^2)^3)$

Solve the middle integral using Maple:

int$(x^2 \cdot y^2 \cdot (1-(x^2+y^2)^3), y=0..\text{sqrt}(1-x^2))$; $\rightarrow$

$\rightarrow \dfrac{2x^2\sqrt{1-x^2}\,(8x^4+20x^2+35)(x^2-1)^2}{315}$

And finally, solve the outer integral using Maple:

int$(\%, x=0..1)$; $\rightarrow \boxed{\dfrac{\pi}{192}}$

6) By converting to polar coordinates, compute:

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2}\, dy\, dx$$

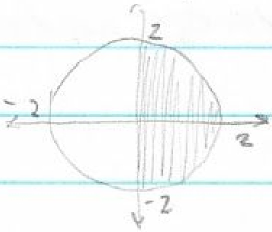
Type of answer: Number

Answer: $\pi\left(\frac{e^4}{2}-\frac{1}{2}\right)$

For converting cartesian to polar:

$x = r\cos(\theta) \qquad y = r\sin(\theta)$

Sketch of original region:



In polar coordinates, our region is a circle of radius 2 from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. So, our new integral is:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{x^2+y^2}\, dy\, dx$$

We need to convert $e^{x^2+y^2}$ to polar. So:

$$e^{x^2+y^2} = e^{r^2\cos(\theta)^2 + r^2\sin(\theta)^2} = e^{r^2(1)} = e^{r^2}$$

dA for polar coordinates is $r\,dr\,d\theta$. So, our final integral is:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 r e^{r^2}\, dr\, d\theta \rightarrow \text{substitute } u = r^2,\ du = 2r\,dr \rightarrow$$

$$\rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \frac{e^u}{2}\, du\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^u}{2}\Big|_0^4 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^4}{2} - \frac{e^0}{2}\, d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^4}{2} - \frac{1}{2}\, d\theta = \theta\left(\frac{e^4}{2}-\frac{1}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(\frac{e^4}{2}-\frac{1}{2}\right) =$$

$$= \boxed{\pi\left(\frac{e^4}{2}-\frac{1}{2}\right)}$$

7) Compute the line integral

$$\int_C 32x^2yz\,ds$$

Where C is the line segment joining (0,0,0) and (2,2,2)

Type of answer: Number

Answer: $\frac{1024\sqrt{3}}{5}$

Our line segment can be represented parametrically as:

$x=t,\ y=t,\ z=t,\ 0\le t\le 2$

The formula for ds in this case is:

$$ds=\sqrt{x_t^2+y_t^2+z_t^2}$$

So first, find all of the derivatives needed:

$x_t=1,\ y_t=1,\ z_t=1$

Plug them into ds:

$ds=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

Next, plug in the parametric representation into the integrated function:

$32x^2yz=32t^2tt=32t^4$

So, our new integral is:

$$\int_0^2 (32\sqrt{3})t^4\,dt=\frac{32\sqrt{3}}{5}t^5\Big|_0^2=\frac{32\sqrt{3}(2^5)}{5}-0=$$

$$=\boxed{\frac{1024\sqrt{3}}{5}}$$

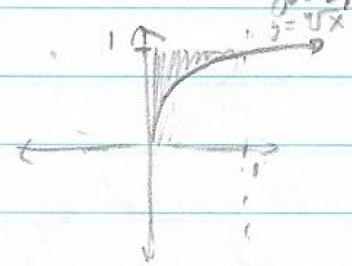
8) Compute:

$$\int_0^1 \int_{\sqrt[4]{x}}^{1} e^{y^5}\, dy\, dx$$

Type of answer: Number

Answer: $\frac{e}{5} - \frac{1}{5}$

Sketch of the region:



We would need to change the order of integration to make a simpler integral.

When we draw a horizontal line from left to right, it passes $x = 0$ first, then $x = y^4$. So, our x-bounds are from 0 to y^4.

Our new y-bounds would be from 0 to 1. So, our new integral is:

$$\int_0^1 \int_0^{y^4} e^{y^5}\, dx\, dy = \int_0^1 x e^{y^5} \Big|_0^{y^4} dy = \int_0^1 y^4 e^{y^5}\, dy \Rightarrow$$

$$\Rightarrow \text{Substitute } u = y^5,\ du = 5y^4 dy \Rightarrow \int_0^1 \frac{e^u}{5}\, du =$$

$$= \frac{e^u}{5}\Big|_0^1 = \frac{e^1}{5} - \frac{e^0}{5} = \boxed{\frac{e}{5} - \frac{1}{5}}$$

9) Compute the volume integral:

$$\iiint_E \frac{10(x^2+y^2+z^2)}{3\pi}\, dV$$

where:

$$E = \{(x,y,z) \mid x^2+y^2+z^2 \le 9\}$$

Type of answer: Number

Answer: 648

Our region is a sphere. So, it would be easier to first convert into spherical coordinates:

$x = r\cos\theta\sin\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi$

Our new region would be:

$E = \{(r,\theta,\phi) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

And our integrated function becomes

$\frac{10(x^2+y^2+z^2)}{3\pi} = \frac{10r^2}{3\pi}$

dV in spherical coordinates is $r^2\sin\phi$. So, our final integral is:

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{3}\frac{10r^4}{3\pi}\sin\phi\, dr\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi}\frac{2r^5}{3\pi}\sin\phi\Big|_0^3\, d\theta\, d\phi =$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{162}{\pi}\sin\phi\, d\theta\, d\phi = \int_0^{\pi}\frac{162\theta}{\pi}\sin\phi\Big|_0^{2\pi}\, d\phi =$$

$$= \int_0^{\pi} 324\sin\phi\, d\phi = -324\cos\phi\Big|_0^{\pi} = -324(-1)+324(1) =$$

$$= \boxed{648}$$

10) Find $\nabla \cdot F$ if:

$$F = \langle e^{xyz}, e^{x^2}, e^{yz^2} \rangle$$

Type of answer: function of x, y, and z

Answer: $yze^{xyz} + 2yze^{yz^2}$

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

So,

$$\nabla \cdot F = \left(\frac{\partial}{\partial x} F[1]\right) + \left(\frac{\partial}{\partial y} F[2]\right) + \left(\frac{\partial}{\partial z} F[3]\right) =$$

$$= \frac{\partial}{\partial x}\left(e^{xyz}\right) + \frac{\partial}{\partial y}\left(e^{x^2}\right) + \frac{\partial}{\partial z}\left(e^{yz^2}\right) =$$

$$= yze^{xyz} + 0 + 2yze^{yz^2} = \boxed{yze^{xyz} + 2yze^{yz^2}}$$