1. (10 pts.)
Find the Jacobian of the transformation from (u,v,w)-space to (x,y,z)-space.

$$y\,\cancel{x} = uv + w \quad , z\,\cancel{y} = uw + v \quad , x\,\cancel{z} = vw + u \quad ,$$

at the point $(u,v,w) =$ ([illegible]). $(3,3,3)$

The **type** of the answers is: number

ans.

-28

$$\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix} = \begin{vmatrix} 1 & w & v \\ v & u & 1 \\ w & 1 & v \end{vmatrix} =$$

$$1(u \cdot v - 1) - w(v^2 - w) + v(v - uw)$$

$$1(3 \cdot 3 - 1) - 3(9 - 3) + 3(3 - 9)$$

$$8 - 18 + (-18) = -28$$

2. (10 points altogether)
(i) (3 points) Show that

$$\mathbf{F} = \langle 3x^2yz+yz+\cos(x+y+z)\ ,\ x^3z+xz+\cos(x+y+z)\ ,\ x^3y+xy+\cos(x+y+z)\rangle\ ,$$

is a conservative vector field.

(ii) (4 point) Find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

(iii) (3 points) Find the line-integral $\int_C \mathbf{F}.dr$ where C is the curve

$$\mathbf{r} = \langle \sin t, \cos t + 1, \sin 2t\rangle \quad , \quad 0 \le t \le \pi \ .$$

$r = \langle 0,0,0\rangle \quad r = \langle 0,2,0\rangle$

The types of the answer is: For (ii) multi fun. For (iii) #

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answers (ii) f(x,y,z)= $x^3yz + xyz + \sin(x+y+z)$

(iii) $-\sin 2$

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i. $\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & y & z \end{vmatrix} = (x^3+x+\cos(x+y+z) - (x^3+x+\cos(x+y+z))i -$
$(3x^2y+y+\cos(x+y+z)) - (3x^2y+y+\cos(x+y+z))j +$
$(3x^2+z+\cos(x+y+z) - (3x^2z+z+\cos(x+y+z))k$

$= 0$ conservative

ii. $f_x = 3x^2yz + yz + \cos(x+y+z) \quad f_y = x^3z + xz + \cos(x+y+z)$
$f_z = x^3y + xy + \cos(x+y+z)$

$f = \int 3x^2yz + yz + \cos(x+y+z)\,dx$
$f = x^3yz + xyz + \sin(x+y+z) + h(y,z) = x^3z + xz + \cos(x+y+z)$
$h(y,z) = 0$

$f = x^3yz + xyz + \sin(x+y+z) + g(z) = x^3y + xy + \cos(x+y+z)$
$f(x,y,z) = x^3yz + xyz + \sin(x+y+z)$

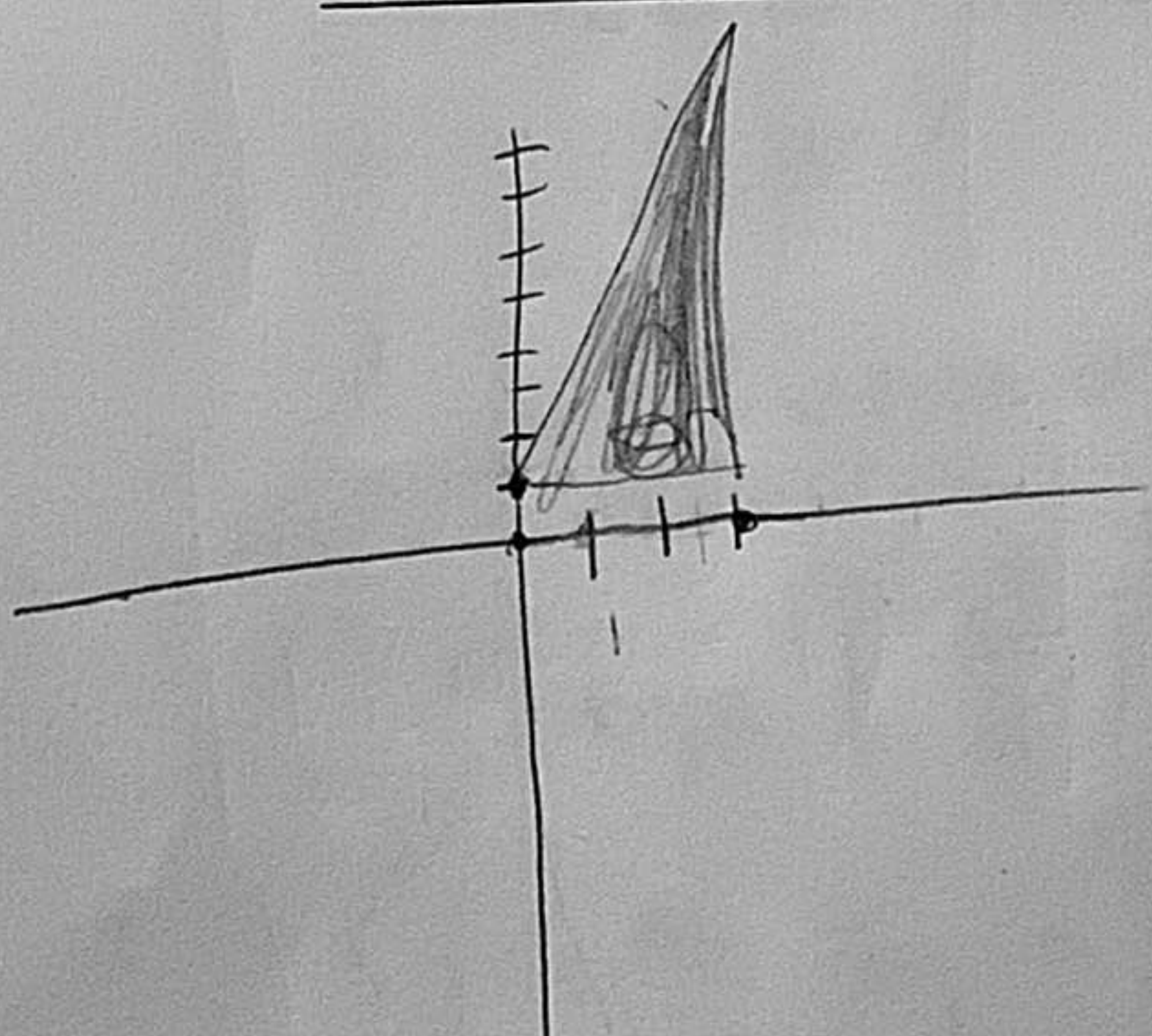
$f(0,0,0) - f(0,2,0) = -\sin 2$

3. (10 points)
Sketch the region of integration and change the order of integration.

$$\int_1^2 \int_0^{e^x+1} F(x,y)\,dy\,dx \qquad \int_0^3 \int_1^{e^x-1} F(x,y)\,dy\,dx$$

The **type** of the answer is: Integral

ans. $$\int_1^{e^3-1} \int_0^{\ln(y+1)} F(x,y)\,dx\,dy$$



$0 \le x \le 3$ $\quad 1 \le y \le e^x - 1$

$0 \le x \le \ln(y+1)$ $\quad y+1 = e^x$

$x = \ln(y+1)$

$1 \le y \le e^3 - 1$

$$\int_1^{e^3-1} \int_0^{\ln(y+1)} F(x,y)\,dx\,dy$$

4. (10 points) Use Largange multipliers (no credit for other methods) to find the smallest value that $x+y+z$ can be, given that $xyz = 1$.

$\lambda x + \lambda y + \lambda z$

The **type** of the answer is: #

ans. 3

$\nabla f = \langle \lambda, \lambda, \lambda \rangle$ $\nabla g = \langle yz, xz, xy \rangle$

$\langle \lambda, \lambda, \lambda \rangle = \lambda \langle yz, xz, xy \rangle$

$\lambda = \lambda yz$, $\lambda = \lambda xz$, $\lambda = \lambda xy$

$8 = (\lambda)^3 (xyz)^2$

$8 = \lambda^3$

$\lambda = \lambda$ $x=1$ $y=1$ $z=1$

$f(1,1,1) = 1+1+1 = 3$

5. (10 points) Compute the volume integral

$$\iiint_E \overset{36}{\cancel{48}}\, x\, y\, z\, dV$$

where E is the region in 3D

$$\{(x, y, z)\,|\, 0 \le x \le y \le z \le 1\}\,.$$

The **type** of the answer is: #

ans. $\frac{3}{4}$

$0 \le z \le 1 \quad 0 \le y \le z \quad 0 \le x \le y$

$$\int_0^1 \int_0^z \int_0^y 36xyz \, dx\, dy\, dz$$

$$36yz\left(\frac{x^2}{2}\Big|_0^y\right) = 18y^3z$$

$$\int_0^z 18y^3z\, dy = \frac{18y^4}{4}z \Big|_0^z = \frac{18z^5}{4}$$

$$\int_0^1 \frac{18z^5}{4}\, dz = \frac{18z^6}{24}\Big|_0^1 = \frac{18}{24} = \frac{3}{4}$$

6. (10 points) By converting to polar coordinates, compute

$$\int_{-\cancel{3}\,4}^{\cancel{3}\,4} \int_{-\sqrt{\cancel{9}\,16-x^2}}^{\sqrt{\cancel{9}\,16-x^2}} \frac{(x^2+y^2)^{\cancel{2}}}{243\pi}\, dy\, dx$$

The **type** of the answer is: #

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ans.

$\frac{512}{972}\pi$

$0 \le r \le 4 \qquad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^4 \frac{r^2}{243\pi}\, r\, dr\, d\theta = \frac{r^4}{972}\Big|_0^4 = \frac{256}{972}$$

$$\int_0^{2\pi} \frac{256}{972}\, d\theta = \frac{512\pi}{972}$$

√

7. (10 points) Compute the line integral

$$\int_C \frac{\cancel{4\sqrt{3}}\,x\,y\,z}{\cancel{3}\,2}\,ds\ ,$$

where C is the line-segment joining $(0,0,0)$ and $\cancel{(1,1,1)}$ $(2,2,2)$

The **type** of the answer is: #

ans.

$\sqrt{12}$

$\langle 0,0,0\rangle + t\langle 2,2,2\rangle = \langle 2t, 2t, 2t\rangle$

$r'(t) = \langle 2,2,2\rangle$

$|r'(t)| = \sqrt{12}$

$\sqrt{12}\int_0^1 \frac{8t^3}{2}\,dt = \sqrt{12}\int_0^1 4t^3\,dt = \frac{4t^4}{4}\Big|_0^1 = \frac{4}{4} = 1$

$\sqrt{12}$

8. (10 points) Compute

$$\int_0^3 \int_{\sqrt{y/3}}^1 e^{x^3}\, dx\, dy \quad .$$

(Hint: Not even Dr. Z. can do $\int e^{x^3}\, dx$, so you must be clever, and first change the order of integration.)

The **type** of the answer is: #

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ans.

$e-1$

$\sqrt{\frac{y}{3}} \le x \le 1$ $\quad$ $0 \le y \le 3$ $\quad$ $\sqrt{\frac{3}{3}} = \sqrt{1}$

$x^2 = \frac{y}{3}$ $\quad$ $0 \le y \le 3x^2$ $\quad$ $\sqrt{\frac{3}{0}} = 0$

$3x^2 = y$ $\quad$ $0 \le x \le 1$

$$\int_0^1 \int_0^{3x^2} e^{x^3}\, dy\, dx = ye^{x^3} = 3x^2 e^{x^3}$$

$$\int_0^1 3x^2 e^{x^3}\, dx =$$

$u = x^3$

$du = 3x^2\, dx$

$$\int du \cdot e^u = e^u = e^{x^3} \Big|_0^1 = e^1 - 1$$

9. (10 points) Compute the volume integral

$$\iiint_E \frac{(x^2+y^2+z^2)}{2\pi}\, dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,x^2+y^2+z^2 \le 1\} \quad .$$

The **type** of the answer(s) is: #

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ans.

$\frac{-2}{5}$

$r = 1$ so $0 \le \rho \le 1$ $\quad 0 \le \phi \le \pi$ $\quad 0 \le \theta \le 2\pi$

$$\frac{1}{2\pi}\int_0^1\int_0^{\pi}\int_0^{2\pi} \rho^2 \rho^2 \sin d\theta\, d\phi\, d\rho$$

$$\frac{1}{2\pi}\left(\theta \rho^4 \sin\phi \Big|_0^{2\pi}\right) = \rho^4 \sin\phi$$

$$\int_0^{\pi} \rho^4 \sin\phi = \rho^4 \cos\phi \Big|_0^{\pi} = \rho^4(-2)$$

$$-2\int_0^1 \rho^4 d\rho = \frac{\rho^5}{5} = \frac{-2}{5}$$

10. (10 points) Find $\nabla \cdot \mathbf{F}$ if

$$\mathbf{F} = \langle \cancel{\sin(xy), \sin(yz)}, \sin(xz) \rangle \quad \cdot F = \langle \cos(xy), \sin(y,z), \cos(xz$$

The **type** of the answer is: multi function

ans.

$\langle -y\sin xy + z\cos yz - x\sin xz$

$$\frac{d}{dx} \cos xy = -\sin xy \cdot y$$

$$\frac{d}{dy} \sin(yz) = \cos yz \cdot z$$

$$\frac{d}{dz} \cos(xz) = -\sin(xz) \cdot x$$

$$\langle -y\sin xy, z\cos yz, -x\sin xz \rangle$$

$$-y\sin xy + z\cos yz - x\sin xz$$