Used Maple For Calculations !! :)

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"Problems from a Previous Final" L1-10 11/22/20

12.5 $r(t) = \langle 1,1,1\rangle + t\langle 1,-1,0\rangle \;;\; (1,0,2)$

$PQ = \langle 1-0, 0-1, 1-1\rangle = \langle 1,-1,0\rangle$

$PR = \langle 1,0,-1\rangle$

$$PQ \times PR = \begin{vmatrix} i & j & K \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = i\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + K\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$$

$= i + j + K \Rightarrow \langle 1,1,1\rangle$

$1(x-1) + 1(y-1) + 1(z-1) = 0 \Rightarrow x+y+z = 3$

13.2 $a(t) = i+j \;;\; v(0) = i-j,\; s(0) = K$

$v(t) = \int a(t)\,dt = \int i+j\,dt = ti + tj + C \quad (C = i-j)$

$r(t) = \int v(t)\,dt = \int ti + tj + i - j\,dt = (\tfrac{1}{2}t^2 + t)i + (\tfrac{1}{2}t^2 - t)j + C$

$v(t) = (t+1)i + (t-1)j \;;\; r(t) = (t^2/2 + t)i + (t^2/2 - t)j + K$

13.3 $r(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle, \quad 0 \le t \le \pi$

(a) $v(t) = \text{deriv.}(r(t)) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t\rangle$

(b) arc length $= \int_0^\pi |r'(t)|\,dt = \sqrt{3}(e^\pi - 1)$

13.4 $r(t) = \langle t, t^2, \tfrac{2}{3}t^3\rangle$ at pt. $(1,1,2/3) \Rightarrow t = 1$

$r'(t) = \langle 1, 2t, 2t^2\rangle \Rightarrow r'(1) = \langle 1,2,2\rangle$

$r''(t) = \langle 0, 2, 4t\rangle \Rightarrow r''(1) = \langle 0,2,4\rangle$

$$r'(1) \times r''(1) = \begin{vmatrix} i & j & K \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} = i\begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} + K\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 4i - 4j + 2K$$

$|r'(1) \times r''(1)| = \sqrt{4^2 + (-4)^2 + 2^2} = \sqrt{16+16+4} = \sqrt{36} = 6$

$|r'(1)| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$$K(t) = \frac{|r'(1) \times r''(1)|}{|r'(1)|^3} = \frac{6}{27} = \frac{2}{9}$$

14.2 $\lim_{(x,y,z)\to(1,1,1)} e^{-xy}\sin(\pi z/2) = e^{-(1)(1)}\sin(\pi(1)/2) = e^{-1}\sin(\pi/2)$

$= e^{-1} = 1/e$

14.4 $z = e^{2x-3y}$ at pt. $(3,2,1)$

$f_x = 2e^{2x-3y} \Rightarrow f_x(3,2) = 2$

$f_y = -3e^{2x-3y} \quad f_y(3,2) = -3$

$(z-1) = 2(x-3) - 3(y-2)$

$z = 2x - 6 - 3y + 6 + 1 \Rightarrow z = 2x - 3y + 1$

14.5 $f(x,y,z) = -x^2+y^2+z^2-1$

(a) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle -2x, 2y, 2z \rangle$

(b) $f(x,y,z)=0$ at pt. $(1,1,1) \Rightarrow z = x-y+1$

(c) $\|\langle 1,2,2\rangle\| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

$\nabla f \cdot u = \langle -2,2,2\rangle \cdot \dfrac{\langle 1,2,2\rangle}{3} = \dfrac{-2+4+4}{3} = \dfrac{6}{3} = 2$

14.6 $f(x,y) = x^3+2xy+y^3$; $x = r-s$ & $y = r+s$

$f_r = (f_x)(x_r)+(f_y)(y_r) = (3x^2+2y)(1)+(2x+3y^2)(1) = 3x^2+2y+2x+3y^2$

$f_s = (f_x)(x_s)+(f_y)(y_s) = (3x^2+2y)(-1)+(2x+3y^2)(1) = -3x^2-2y+2x+3y^2$

$df/dr = 3(r-s)^2+2(r+s)+3(r+s)^2+2(r-s)$

$df/ds = -3(r-s)^2-2(r+s)+3(r+s)^2+2(r-s)$

14.6 $\sin(x+2y+3z) = 5xyz+1 \Rightarrow F(x,y,z) = \sin(x+2y+3z)-5xyz-1$

$\dfrac{dz}{dx} = -\dfrac{F_x}{F_z} = \dfrac{(5yz-\cos(x+2y+3z))}{(3\cos(x+2y+3z)-5xy)}$

$\dfrac{dz}{dy} = -\dfrac{F_y}{F_z} = \dfrac{(5xz-2\cos(x+2y+3z))}{(3\cos(x+2y+3z)-5xy)}$

14.7 $f(x,y) = 4x^2+y^2+2x^2y-1$

$f_x(x,y) = 8x+4xy = 0$ (1)

$f_y(x,y) = 2y+2x^2 = 0$ (2) $y = -x^2$ sub.

$8x+4x(-x^2) = 0$	$y = -(0)^2 = 0$
$8x-4x^3 = 0$	$y = -(\sqrt{2})^2 = -2$
$4x(2-x^2) = 0$	$y = -(-\sqrt{2})^2 = -2$
$x = 0, \pm\sqrt{2}$	Crit. pts. : $(0,0)$, $(\sqrt{2},-2)$, $(-\sqrt{2},-2)$
$f_{xx}(x,y) = 8+4y$	$D(x,y) = f_{xx}f_{yy}-f_{xy}^2 = (8+4y)(2)-(4x)^2$
$f_{yy}(x,y) = 2$	$= 16+8y-16x^2 = -16x^2+8y+16$
$f_{xy}(x,y) = 4x$	

$D(0,0) = 16 > 0 \Rightarrow f_{xx}(0,0) = 8 > 0$ (local min.) $\Rightarrow f(0,0) = -1$

$D(\sqrt{2},-2) = -32 < 0 \Rightarrow$ saddle pt.

$D(-\sqrt{2},-2) = -32 < 0 \Rightarrow$ saddle pt.

14.7 $f(x,y) = 6y^2-2y^3+3x^2+6xy$

Local max : none. Local Min: location: $(0,0)$, value: 0. Saddle pt: $(-1,1)$

Book Exercises (2 per section) L1-10 11/22/20

12.1: # 43, 49

43. Vector of length 4 in the direction of $u = \langle -1, -1 \rangle$

$$e_u = \frac{1}{\|u\|} u = \frac{1}{\sqrt{(-1)^2+(-1)^2}} \langle -1,-1 \rangle = \frac{1}{\sqrt{2}} \langle -1,-1\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$4e_u = 4\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{4}{\sqrt{2}}, -\frac{4}{\sqrt{2}} \right\rangle = \langle -2\sqrt{2}, -2\sqrt{2} \rangle$$

49. Find all scalars λ s.t. $\lambda(2,3)$ has length 1.

$$\|\lambda\langle 2,3\rangle\| = |\lambda| \|\langle 2,3 \rangle\| = |\lambda|\sqrt{2^2+3^2} = |\lambda|\sqrt{13}$$

$$|\lambda|\sqrt{13} = 1 \qquad \Rightarrow \quad \lambda_1 = \frac{1}{\sqrt{13}}, \quad \lambda_2 = -\frac{1}{\sqrt{13}}$$
$$|\lambda| = 1/\sqrt{13}$$

12.2: # 37, 39

37. Passes through $(1,1,1)$ & $(3,-5,2)$

$r(t) = (1-t)\vec{OP} + t\vec{OQ}$; $\vec{OP} = \langle 1,1,1\rangle$ & $\vec{OQ} = \langle 3,-5,2 \rangle$

$r(t) = (1-t)\langle 1,1,1\rangle + t\langle 3,-5,2\rangle = \langle 1+2t, 1-6t, 1+t \rangle$

39. Passes through O & $(4,1,1)$

$r(t) = (1-t)\langle 0,0,0\rangle + t\langle 4,1,1\rangle = \langle 0,0,0\rangle + \langle 4t, t, t\rangle = \langle 4t, t, t\rangle$

12.3: # 61, 75

61. $u = \langle 3,2,1\rangle$, $v = \langle 1,0,1\rangle$

$u \cdot v = 4$; $v \cdot v = \|v\|^2 = 1^2 + 1^2 = 2$ $\Rightarrow$ $\left\| \left(\frac{u\cdot v}{v \cdot v}\right) v \right\| = \frac{4}{2}\|v\| = 2\sqrt{2}$

75. Calc. projection of $\vec{AC}$ along $\vec{AD}$: $\overline{DC}$ is $\perp$ to the face OAD of the cube. Hence, it is orthogonal to the segment $\overline{AD}$ on this face. Therefore, the projection of the vector $\vec{AC}$ along $\vec{AD}$ is the vector $\vec{AD}$ itself.

12.4: # 57, 73

57. Show that 3 pts. P, Q, R are collinear if & only if $\vec{PQ} \times \vec{PR} = 0$.
They lie on 1 line if $\vec{PQ}$ & $\vec{PR}$ are parallel. By basic props. of cross product $\Rightarrow$ $\vec{PQ} \times \vec{PR} = 0$

73. Prove that $v \times w = v \times u$ if & only if $u = w + \lambda v$ for some scalar λ. $(v \neq 0)$

$v \times w = v \times u$ $u - w = \lambda v$

$\cancel{0 = v\times u - v \times w}$

$u = w + \lambda v$

$\cancel{0 = v \times (u - w)}$

12.5 : # 43, 45

43. $3x - 9y + 4z = 5$, yz

The yz-plane has the eq. $x = 0$, hence the intersection of the plane w/ the yz-plane must satisfy both $x = 0$ & the eq. of the plane $3x - 9y + 4z = 5$. That is, this is the set of all pts. $(0, y, z)$ in the yz-plane s.t. $-9y + 4z = 5$.

45. $3x + 4z = -2$, xy

The trace of the plane $3x + 4z = -2$ in the xy coordinate plane is the set of all pts. that satisfy the eq. of the plane & the eq. $z = 0$ of the xy coordinate plane. Thus, we sub. $z = 0$ in $3x + 4z = -2$ to obtain the line $3x = -2$ or $x = -2/3$ in the xy-plane.

13.1 : # 15, 19

15. (A) => (ii) ; (C) => (i) ; (B) => (iii)

19. $r(t) = \langle \sin t, 0, 4 + \cos t \rangle$ => $x(t) = \sin t$, $z(t) = 4 + \cos t$

$x^2 + (z-4)^2 = \sin^2 t + \cos^2 t = 1$ (traces ○ of radius 1, centered at (0,0,4))

13.2 : #51, 53

51. $r''(t) = 16k$, $r(0) = \langle 1, 0, 0 \rangle$, $r'(0) = \langle 0, 1, 0 \rangle$

$r'(t) = \int r''(t)\,dt = \int 16k\,dt = (16t)k + c_1$

$r(t) = \int r'(t)\,dt = \int ((16t)k + c_1)\,dt = (\int 16(t)\,dt)k + c_1 t + c_2 = (8t^2)k + c_1 t + c_2$

$r'(0) = c_1 = \langle 0, 1, 0 \rangle = j$

$r(0) = 0k + c_1 \cdot 0 + c_2 = \langle 1, 0, 0 \rangle$ => $c_2 = \langle 1, 0, 0 \rangle = i$

$r(t) = (8t^2)k + tj + i = i + tj + (8t^2)k$

53. $r''(t) = \langle 0, 2, 0 \rangle$, $r(3) = \langle 1, 1, 0 \rangle$, $r'(3) = \langle 0, 0, 1 \rangle$

$r'(t) = \int r''(t)\,dt = \int \langle 0, 2, 0 \rangle\,dt = \langle 0, 2t, 0 \rangle + c_1$

$r(t) = \int r'(t)\,dt = \int (\langle 0, 2t, 0 \rangle + c_1)\,dt = \langle 0, t^2, 0 \rangle + c_1 t + c_2$

$r'(3) = \langle 0, 6, 0 \rangle + c_1 = \langle 0, 0, 1 \rangle$ => $c_1 = \langle 0, -6, 1 \rangle$

$r(3) = \langle 0, 9, 0 \rangle + c_1(3) + c_2 = \langle 1, 1, 0 \rangle$

$\langle 0, 9, 0 \rangle + \langle 0, -18, 3 \rangle + c_2 = \langle 1, 1, 0 \rangle$ => $c_2 = \langle 1, 10, -3 \rangle$

$r(t) = \langle 0, t^2, 0 \rangle + t\langle 0, -6, 1 \rangle + \langle 1, 10, -3 \rangle$

$= \langle 1, t^2 - 6t + 10, t - 3 \rangle$

13.3 : # 17 , 19

17. $y = x^2$; at pt. $(1,1)$, speed $= 500$ km/h

$m = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{d}{dx}(x^2)\right|_{x=1} = 2x\,\big|_{x=1} = 2 \Rightarrow \langle 1,2\rangle$ is direction vector of tan. line at $x=1$

$500 = \|\lambda\langle 1,2\rangle\| = |\lambda|\,\|\langle 1,2\rangle\| = \sqrt{5}\,|\lambda| \Rightarrow \lambda = \pm 100\sqrt{5}$

Tan. vector : $100\sqrt{5}\langle 1,2\rangle$

19. $\int_0^T r'(u)\,du = r(T) - r(0) \Rightarrow r(T) = r(0)$

$\int_0^T \|r'(u)\|\,du$ = length of the path traveled by the bee in the time interval $0 \le t \le T$.

13.4 : # 29 , 31

29. $\langle t^2, t^3\rangle$, $t=2$ $\Rightarrow x(t) = t^2$, $y(t) = t^3$

$x'(t) = 2t$, $y'(t) = 3t^2$ $\Rightarrow$ $x'(2) = 4$, $y'(2) = 12$

$x''(t) = 2$, $y''(t) = 6t$ $\qquad$ $x''(2) = 2$, $y''(2) = 12$

$$K(2) = \frac{|x'(2)y''(2) - x''(2)y'(2)|}{(x'(2)^2 + y'(2)^2)^{3/2}} = \frac{|4\cdot 12 - 2\cdot 12|}{(4^2+12^2)^{3/2}} = \frac{24}{160^{3/2}} \approx 0.012$$

31. $\langle t\cos t, \sin t\rangle$, $t = \pi$ $\Rightarrow x(t) = t\cos t$, $y(t) = \sin t$

$x'(\pi) = -1$, $y'(\pi) = -1$

$x''(\pi) = \pi$, $y''(\pi) = 0$

$$K(\pi) = \frac{|x'(\pi)y''(\pi) - x''(\pi)y'(\pi)|}{(x'(\pi)^2 + y'(\pi)^2)^{3/2}} = \frac{|-1\cdot 0 - \pi\cdot(-1)|}{((-1)^2 + (-1)^2)^{3/2}} = \frac{\pi}{2\sqrt{2}} \approx 1.11$$

13.5 : # 27 , 29

27. $F = \langle 5,2\rangle$ acts on a 10-kg mass ; find $r(10)$ if $v_0 = \langle 2,-3\rangle$

$F = ma \Rightarrow a = \langle 0.5, 0.2\rangle \Rightarrow v = \langle 0.5t + 2, 0.2t - 3\rangle$

$\Rightarrow r = \langle 0.25t^2 + 2t, 0.1t^2 - 3t\rangle$

$\Rightarrow r(10) = \langle 45, -20\rangle$

29. (a) if avg. velocity $= 0$, then particle must be back at $t = T$

$\Rightarrow \bar{v} = 1/T \int_0^T r'(t)\,dt = r(t)\big|_0^T$

(b) avg. speed need not $\neq 0$! Ex: $r(t) = \langle \cos t, \sin t\rangle$. from 0 to 2π, avg. velocity $= 0$, but constant avg. speed $= 1$

14.1 : #45, 47

45. Δd (B to C) = 0.0005 Kg/m^3 ; Δs (B to C) = 33.6 − 32.7 = 0.9 ppt.
Avg. ROC from B to C = $\Delta d/\Delta s$ = 0.0005/0.9 = 0.000556 Kg/m^3 • ppt

47. The 2 adjacent level curves are closer to the level curve of A than the corresponding 2 adjacent level curves are to the level curve of B. This suggest that water density is more sensitive to a change in temp. at A than B.

14.2 : # 33, 37

33. $\lim\limits_{(x,y)\to(\pi,0)} \dfrac{\cos x}{\sin y} = \text{DNE}$

37. $\lim\limits_{(x,y)\to(-3,-2)} (x^2y^3 + 4xy) = 9(-8) + 4(-3)(-2) = -72 + 24 = -48$

14.3 : # 41, 43

41. $f(x,y) = 3x^2y + 4x^3y^2 - 7xy^5$, $f_x(1,2) = 6 \cdot 1 \cdot 2 + 12 \cdot 1^2 \cdot 2^2 - 7 \cdot 2^5 = -164$
$f_x(x,y) = 6xy + 12x^2y^2 - 7y^5$

43. $g(u,v) = u\ln(u+v)$, $g_u(1,2) = \ln(1+2) + 1/(1+2) = \ln 3 + 1/3$
$g_u(u,v) = d/du\,(u\ln(u+v)) = 1 \cdot \ln(u+v) + u \cdot 1/(u+v) = \ln(u+v) + u/(u+v)$

14.4 : # 29, 31

29. $z = f(x,y)$ at $(-2,3,4)$ has eq. $4x + 2y + z = 2 \iff z = 2 - 4x - 2y$
$f(-2.1, 3.1) \approx 2 - 4(-2.1) - 2(3.1) = 4.2$

31. $W = 34$ Kg & $H = 1.3$ m
$I(34+h, 1.3+K) - I(34, 1.3) \approx dI/dW\,(34, 1.3)h + dI/dH\,(34, 1.3)K$
$\Delta I \approx dI/dW\,(34, 1.3) \cdot 2 + dI/dH\,(34, 1.3) \cdot 0.02$
$dI/dW = d/dW\; W/H^2 = 1/H^2 \Rightarrow dI/dW\,(34, 1.3) = 0.5917$
$dI/dH = W\, d/dH\; H^{-2} = W \cdot (-2H^{-3}) = -2W/H^3 \Rightarrow dI/dH\,(34, 1.3) = -30.9513$
$\Delta I \approx 0.5917 \cdot 2 - 30.9513 \cdot 0.02 = 0.5644$

14.5 : # 53, 57

53. Find a fn. $f(x,y,z)$ s.t. $\nabla f = \langle z, 2y, x\rangle$
$f(x,y,z) = xz + y^2$

57. Est. $\Delta f = f(3.53, 8.98) - f(3.5, 9)$ assuming that $\nabla f_{(3.5,9)} = \langle 2, -1 \rangle$
$\Delta f \approx \nabla f_P \cdot \Delta v \quad \Rightarrow \quad \Delta v = \langle 3.53 - 3.5,\ 8.98 - 9 \rangle = \langle 0.03, -0.02 \rangle$
$\approx \nabla f_{(3,5,9)} \cdot \Delta v = \langle 2, -1 \rangle \cdot \langle 0.03, -0.02 \rangle = 0.08$

14.6 : 35, 43

35. Compute $\nabla(1/r) \Rightarrow$ let $F(r) = 1/r \Rightarrow F'(r) = -1/r^2$
$\nabla(1/r) = F'(r)\, e_r = -1/r^2 \cdot r/\|r\| = -1/r^3 \cdot r$

43. Show that $f(x)$ is differentiable & $c \neq 0$ is a constant, then $u(x,t) = f(x-ct)$ satisfies the so-called advection eq.

$du/dt + c\, du/dx = 0 \qquad s = x - ct, \quad u(x,t) = f(s)$

$du/dt = f'(s)\, ds/dt = f'(s) \cdot (-c) = -c f'(s)$
$du/dx = f'(s)\, ds/dx = f'(s) \cdot 1 = f'(s)$
$du/dt = -c\, du/dx$ or $du/dt + c\, du/dx = 0$

14.7 : # 33, 43

33. The largest & smallest vals. of f on the closed square $0 \le x, y \le 1$ are $f(1,1) = 2$ & $f(-1,-1) = -2$. However, on the open square $0 < x, y < 1$, f can never attain these max. & min. values, since the boundary (& in particular pts. $(1,1)$ & $(-1,-1)$) is not included in the domain. This doesn't contradict Thm. 3 since the domain is open.

43. $f(x,y) = x^2 + 2y^2, \quad 0 \le x \le 1, \quad 0 \le y \le 1$
$f(x,y)$ is max. at $(1,1)$
is min. if $x = y = 0 \quad (0,0)$
Global max $= f(1,1) = 1^2 + 2 \cdot 1^2 = 3$
Global min $= f(0,0) = 0^2 + 2 \cdot 0^2 = 0$