A vector field in $\mathbf{R}^2$ is represented by a pair of functions $\mathbf{F} = \langle F_1, F_2 \rangle$. We al\ assume that the components F_j are smooth functions on their domains.

- The *divergence* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is the scalar function given by

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- The *curl* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is the vector field given by

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

- If $\mathbf{F} = \nabla f$, then f is called a *potential function* for $\mathbf{F}$.
- $\mathbf{F}$ is called *conservative* if it has a potential function.
- Any two potential functions for a conservative vector field differ by a constant (on an open, connected domain).
- A conservative vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ satisfies the condition:

$$\operatorname{curl}(\mathbf{F}) = \mathbf{0}, \quad \text{or equivalently,} \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

- We define

$$r = \sqrt{x^2 + y^2 + z^2}$$

- The radial unit vector field and the inverse-square vector field are conservative:

$$\mathbf{e}_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \nabla r, \qquad \frac{\mathbf{e}_r}{r^2} = \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle = \nabla(-r^{-1})$$

16.1 EXERCISES

Preliminary Questions

1. Which of the following is a unit vector field in the plane?

(a) $\mathbf{F} = \langle y, x \rangle$

(b) $\mathbf{F} = \left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$

(c) $\mathbf{F} = \left\langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

2. Sketch an example of a nonconstant vector field in the plane in which each vector is parallel to $\langle 1, 1 \rangle$.

3. Show that the vector field $\mathbf{F} = \langle -z, 0, x \rangle$ is orthogonal to the position vector $\overrightarrow{OP}$ at each point P. Give an example of another vector field with this property.

4. Give an example of a potential function for $\langle yz, xz, xy \rangle$ other than $f(x, y, z) = xyz$.

Exercises

1. Compute and sketch the vector assigned to the points $P = (1, 2)$ and $Q = (-1, -1)$ by the vector field $\mathbf{F} = \langle x^2, x \rangle$.

2. Compute and sketch the vector assigned to the points $P = (1, 2)$ and $Q = (-1, -1)$ by the vector field $\mathbf{F} = \langle -y, x \rangle$.

3. Compute and sketch the vector assigned to the points $P = (0, 1, 1)$ and $Q = (2, 1, 0)$ by the vector field $\mathbf{F} = \langle xy, z^2, x \rangle$.

4. Compute the vector assigned to the points $P = (1, 1, 0)$ and $Q = (2, 1, 2)$ by the vector fields $\mathbf{e}_r$, $\frac{\mathbf{e}_r}{r}$, and $\frac{\mathbf{e}_r}{r^2}$.

In Exercises 5–12, sketch the following planar vector fields by drawing the vectors attached to points with integer coordinates in the rectangle $-3 \le x \le 3$, $-3 \le y \le 3$. Instead of drawing the vectors with their true lengths, scale them if necessary to avoid overlap.

5. $\mathbf{F} = \langle 1, 0 \rangle$

6. $\mathbf{F} = \langle 1, 1 \rangle$

7. $\mathbf{F} = x\mathbf{i}$

8. $\mathbf{F} = y\mathbf{i}$

9. $\mathbf{F} = \langle 0, x \rangle$

10. $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j}$

11. $\mathbf{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$

12. $\mathbf{F} = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$

ercises 13–16, match each of the following planar vector fields the corresponding plot in Figure 12.

$\mathbf{F} = \langle 2, x \rangle$

14. $\mathbf{F} = \langle 2x + 2, y \rangle$

5. $\mathbf{F} = \langle y, \cos x \rangle$

16. $\mathbf{F} = \langle x + y, x - y \rangle$

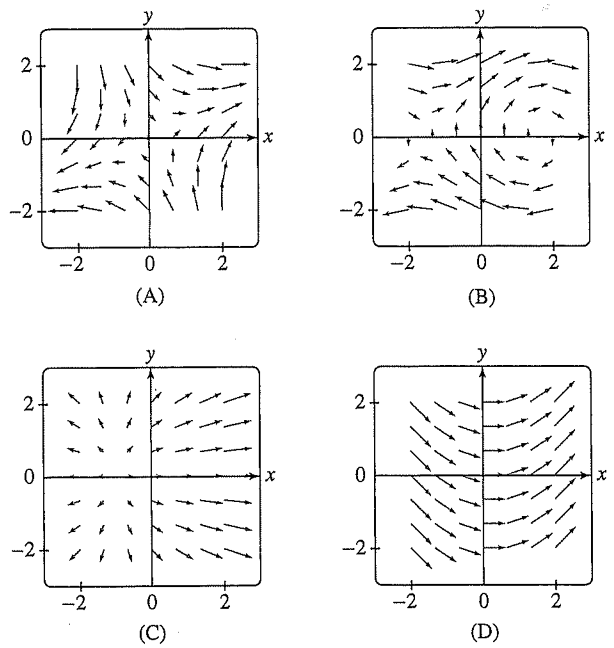


FIGURE 12

In Exercises 17–20, match each three-dimensional vector field with the corresponding plot in Figure 13.

17. $\mathbf{F} = \langle 1, 1, 1 \rangle$

18. $\mathbf{F} = \langle x, 0, z \rangle$

19. $\mathbf{F} = \langle x, y, z \rangle$

20. $\mathbf{F} = \mathbf{e}_r$

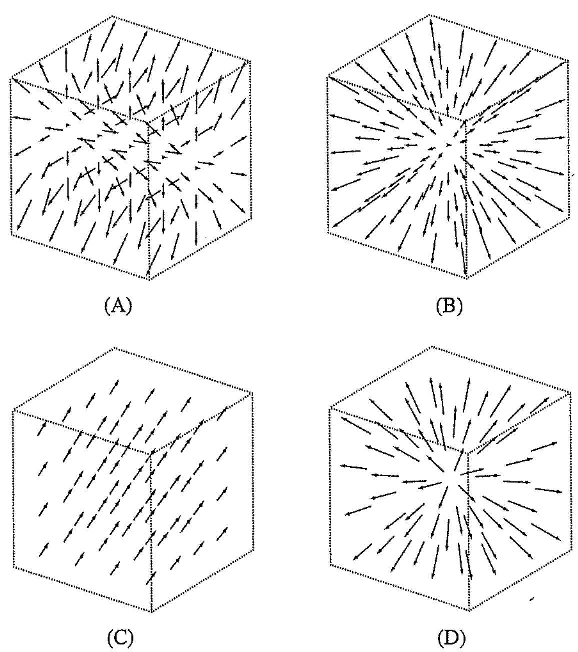


FIGURE 13

21. A river 200 meters wide is modeled by the region in the xy-plane given by $-100 \le x \le 100$. The velocity vector field on the surface of the river is given by $\mathbf{F} = \left\langle \frac{-x}{20}, 20 - \frac{x^2}{1000} \right\rangle$ in meters per second (m/s). Determine the coordinates of those points that have the maximum speed.

22. The velocity vectors in kilometers per hour for the wind speed of a tornado near the ground are given by the vector field $\mathbf{F} = \left\langle \frac{-y}{e^{(x^2+y^2-1)^2}}, \frac{x}{e^{(x^2+y^2-1)^2}} \right\rangle$. Determine the coordinates of those points where the wind speed is the highest.

In Exercises 23–30, calculate $\text{div}(\mathbf{F})$ *and* $\text{curl}(\mathbf{F})$.

23. $\mathbf{F} = \langle xy, yz, y^2 - x^3 \rangle$

24. $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

25. $\mathbf{F} = \langle x - 2zx^2, z - xy, z^2x^2 \rangle$

26. $\sin(x + z)\mathbf{i} - ye^{xz}\mathbf{k}$

27. $\mathbf{F} = \langle z - y^2, x + z^3, y + x^2 \rangle$

28. $\mathbf{F} = \left\langle \frac{y}{x}, \frac{y}{z}, \frac{z}{x} \right\rangle$

29. $\mathbf{F} = \langle e^y, \sin x, \cos x \rangle$

30. $\mathbf{F} = \left\langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, 0 \right\rangle$

In Exercises 31–37, prove the identities assuming that the appropriate partial derivatives exist and are continuous.

31. $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div}(\mathbf{F}) + \text{div}(\mathbf{G})$

32. $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl}(\mathbf{F}) + \text{curl}(\mathbf{G})$

33. $\text{div}\,\text{curl}(\mathbf{F}) = 0$

34. $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}(\mathbf{F}) - \mathbf{F} \cdot \text{curl}(\mathbf{G})$

35. If f is a scalar function, then $\text{div}(f\mathbf{F}) = f\,\text{div}(\mathbf{F}) + \mathbf{F} \cdot \nabla f$.

36. $\text{curl}(f\mathbf{F}) = f\,\text{curl}(\mathbf{F}) + (\nabla f) \times \mathbf{F}$

37. $\text{div}(\nabla f \times \nabla g) = 0$

38. Find (by inspection) a potential function for $\mathbf{F} = \langle x, 0 \rangle$ and prove that $\mathbf{G} = \langle y, 0 \rangle$ is not conservative.

In Exercises 39–45, find a potential function for the vector field $\mathbf{F}$ *by inspection or show that one does not exist.*

39. $\mathbf{F} = \langle x, y \rangle$

40. $\mathbf{F} = \langle yz, xz, y \rangle$

41. $\mathbf{F} = \langle ye^{xy}, xe^{xy} \rangle$

42. $\mathbf{F} = \langle 2xyz, x^2z, x^2yz \rangle$

43. $\mathbf{F} = \langle yz^2, xz^2, 2xyz \rangle$

44. $\mathbf{F} = \langle 2xze^{x^2}, 0, e^{x^2} \rangle$

45. $\mathbf{F} = \langle yz\cos(xyz), xz\cos(xyz), xy\cos(xyz) \rangle$.

46. Find potential functions for $\mathbf{F} = \frac{\mathbf{e}_r}{r^3}$ and $\mathbf{G} = \frac{\mathbf{e}_r}{r^4}$ in $\mathbf{R}^3$. *Hint:* See Example 8.

47. Show that $\mathbf{F} = \langle 3, 1, 2 \rangle$ is conservative. Then prove more generally that any constant vector field $\mathbf{F} = \langle a, b, c \rangle$ is conservative.

48. Let $\varphi = \ln r$, where $r = \sqrt{x^2 + y^2}$. Express $\nabla\varphi$ in terms of the unit radial vector $\mathbf{e}_r$ in $\mathbf{R}^2$.

49. For $P = (a, b)$, we define the unit radial vector field based at P:

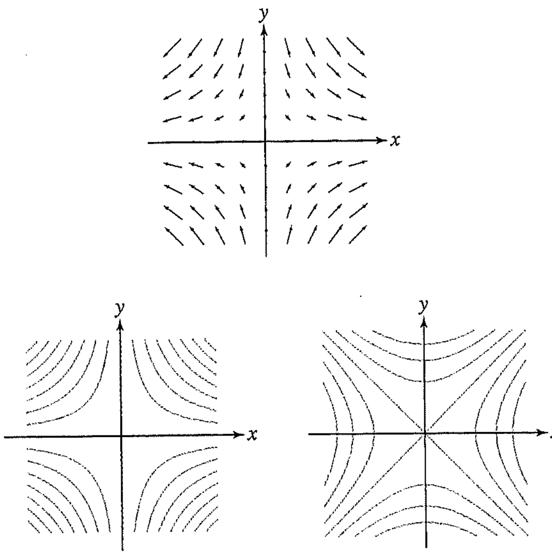
$$\mathbf{e}_P = \frac{\langle x - a, y - b \rangle}{\sqrt{(x - a)^2 + (y - b)^2}}$$

(a) Verify that $\mathbf{e}_P$ is a unit vector field.

(b) Calculate $\mathbf{e}_P(1, 1)$ for $P = (3, 2)$.

(c) Find a potential function for $\mathbf{e}_P$.

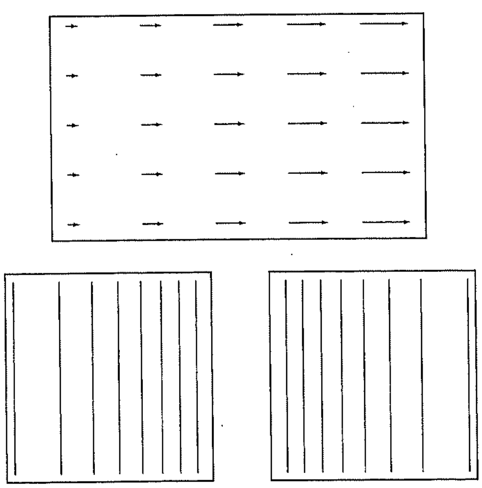
50. Which of (A) or (B) in Figure 14 is the contour plot of a potential function for the vector field **F**? Recall that the gradient vectors are perpendicular to the level curves.



(A) (B)

FIGURE 14

51. Which of (A) or (B) in Figure 15 is the contour plot of a potential function for the vector field **F**?



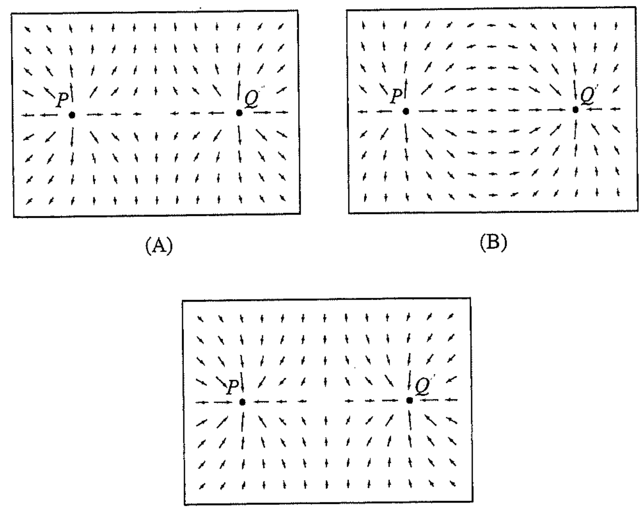
(A) (B)

FIGURE 15

52. Match each of these descriptions with a vector field in Figure

(a) The gravitational field created by two planets of equal mass loc at P and Q

(b) The electrostatic field created by two equal and opposite charg located at P and Q (representing the force on a negative test charge opposite charges attract and like charges repel)



(C)

FIGURE 16

53. In this exercise, we show that the vector field **F** in Figure 17 is not conservative. Explain the following statements:

(a) If a potential function f for **F** exists, then the level curves of f must be vertical lines.

(b) If a potential function f for **F** exists, then the level curves of f must grow farther apart as y increases.

(c) Explain why (a) and (b) are incompatible, and hence f cannot exist.

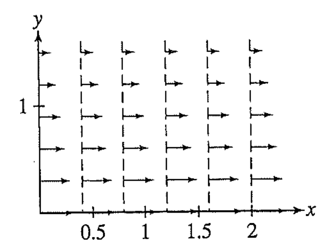


FIGURE 17

Further Insights and Challenges

54. Show that any vector field of the form

$$\mathbf{F} = \langle f(x), g(y), h(z) \rangle$$

has a potential function. Assume that f, g, and h are continuous.

55. Let $\mathcal{D}$ be a disk in $\mathbf{R}^2$. This exercise shows that if

$$\nabla f(x, y) = \mathbf{0}$$

for all (x, y) in $\mathcal{D}$, then f is constant. Consider points $P = (a, b)$, $Q = (c, d)$, and $R = (c, b)$ as in Figure 18.

(a) Use single-variable calculus to show that f is constant along the segments $\overline{PR}$ and $\overline{RQ}$.

(b) Conclude that $f(P) = f(Q)$ for any two points $P, Q \in \mathcal{D}$.

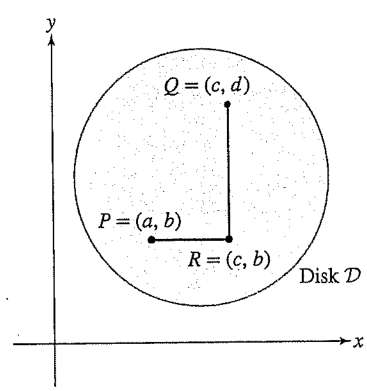


FIGURE 18

Exercises

1. Let $f(x, y, z) = x + yz$, and let C be the line segment from $P = (0, 0, 0)$ to $(6, 2, 2)$.

(a) Calculate $f(\mathbf{r}(t))$ and $ds = \|\mathbf{r}'(t)\|\, dt$ for the parametrization $\mathbf{r}(t) = (6t, 2t, 2t)$ for $0 \le t \le 1$.

(b) Evaluate $\int_C f(x, y, z)\, ds$.

2. Repeat Exercise 1 with the parametrization $\mathbf{r}(t) = (3t^2, t^2, t^2)$ for $0 \le t \le \sqrt{2}$.

3. Let $\mathbf{F} = \langle y^2, x^2 \rangle$, and let C be the curve $y = x^{-1}$ for $1 \le x \le 2$, oriented from left to right.

(a) Calculate $\mathbf{F}(\mathbf{r}(t))$ and $d\mathbf{r} = \mathbf{r}'(t)\, dt$ for the parametrization of C given by $\mathbf{r}(t) = (t, t^{-1})$.

(b) Calculate the dot product $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

4. Let $\mathbf{F}(x, y, z) = \langle z^2, x, y \rangle$, and let C be the curve that is given by $\mathbf{r}(t) = \langle 3 + 5t^2, 3 - t^2, t \rangle$ for $0 \le t \le 2$.

(a) Calculate $\mathbf{F}(\mathbf{r}(t))$ and $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

(b) Calculate the dot product $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

In Exercises 5–8, compute the integral of the scalar function or vector field over $\mathbf{r}(t) = (\cos t, \sin t, t)$ *for* $0 \le t \le \pi$.

5. $f(x, y, z) = x^2 + y^2 + z^2$

6. $f(x, y, z) = xy + z$

7. $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$

8. $\mathbf{F}(x, y, z) = \langle xy, 2, z^3 \rangle$

In Exercises 9–16, compute $\int_C f\, ds$ *for the curve specified.*

9. $f(x, y) = \sqrt{1 + 9xy}$, $y = x^3$ for $0 \le x \le 1$

10. $f(x, y) = \dfrac{y^3}{x^7}$, $y = \frac{1}{4}x^4$ for $1 \le x \le 2$

11. $f(x, y, z) = z^2$, $\mathbf{r}(t) = (2t, 3t, 4t)$ for $0 \le t \le 2$

12. $f(x, y, z) = 3x - 2y + z$, $\mathbf{r}(t) = (2 + t, 2 - t, 2t)$ for $-2 \le t \le 1$

13. $f(x, y, z) = xe^{z^2}$, piecewise linear path from $(0, 0, 1)$ to $(0, 2, 0)$ to $(1, 1, 1)$

14. $f(x, y, z) = x^2 z$, $\mathbf{r}(t) = (e^t, \sqrt{2}t, e^{-t})$ for $0 \le t \le 1$

15. $f(x, y, z) = 2x^2 + 8z$, $\mathbf{r}(t) = (e^t, t^2, t)$, $0 \le t \le 1$

16. $f(x, y, z) = 6xz - 2y^2$, $\mathbf{r}(t) = \left(t, \dfrac{t^2}{\sqrt{2}}, \dfrac{t^3}{3}\right)$, $0 \le t \le 2$

17. Calculate $\int_C 1\, ds$, where the curve C is parametrized by $\mathbf{r}(t) = (4t, -3t, 12t)$ for $2 \le t \le 5$. What does this integral represent?

18. Calculate $\int_C 1\, ds$, where the curve C is parametrized by $\mathbf{r}(t) = (e^t, \sqrt{2}t, e^{-t})$ for $0 \le t \le 2$.

In Exercises 19–26, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ *for the oriented curve specified.*

19. $\mathbf{F}(x, y) = \langle x^2, xy \rangle$, line segment from $(0, 0)$ to $(2, 2)$

20. $\mathbf{F}(x, y) = \langle 4, y \rangle$, quarter circle $x^2 + y^2 = 1$ with $x \le 0$, $y \le 0$, oriented counterclockwise

21. $\mathbf{F}(x, y) = \langle x^2, xy \rangle$, part of circle $x^2 + y^2 = 9$ with $x \le 0$, $y \ge 0$, oriented clockwise

22. $\mathbf{F}(x, y) = \langle e^{y-x}, e^{2x} \rangle$, piecewise linear path from $(1, 1)$ to $(2, 2)$ to $(0, 2)$

23. $\mathbf{F}(x, y) = \langle 3zy^{-1}, 4x, -y \rangle$, $\mathbf{r}(t) = (e^t, e^t, t)$ for $-1 \le t \le 1$

24. $\mathbf{F}(x, y) = \left\langle \dfrac{-y}{(x^2 + y^2)^2}, \dfrac{x}{(x^2 + y^2)^2} \right\rangle$, circle of radius R with center at the origin oriented counterclockwise

25. $\mathbf{F}(x, y, z) = \left\langle \dfrac{1}{y^3 + 1}, \dfrac{1}{z + 1}, 1 \right\rangle$, $\mathbf{r}(t) = (t^3, 2, t^2)$ for $0 \le t \le 1$

26. $\mathbf{F}(x, y, z) = \langle z^3, yz, x \rangle$, quarter of the circle of radius 2 in the yz-plane with center at the origin where $y \ge 0$ and $z \ge 0$, oriented clockwise when viewed from the positive x-axis

In Exercises 27–32, evaluate the line integral.

27. $\int_C y\, dx - x\, dy$, parabola $y = x^2$ for $0 \le x \le 2$

28. $\int_C y\, dx + z\, dy + x\, dz$, $\mathbf{r}(t) = (2 + t^{-1}, t^3, t^2)$ for $0 \le t \le 1$

29. $\int_C (x - y)\, dx + (y - z)\, dy + z\, dz$, line segment from $(0, 0, 0)$ to $(1, 4, 4)$

30. $\int_C z\, dx + x^2\, dy + y\, dz$, $\mathbf{r}(t) = (\cos t, \tan t, t)$ for $0 \le t \le \frac{\pi}{4}$

31. $\int_C \dfrac{-y\, dx + x\, dy}{x^2 + y^2}$, segment from $(1, 0)$ to $(0, 1)$.

32. $\int_C y^2\, dx + z^2\, dy + (1 - x^2)\, dz$, quarter of the circle of radius 1 in the xz-plane with center at the origin in the quadrant $x \ge 0$, $z \le 0$, oriented counterclockwise when viewed from the positive y-axis

33. CAS Let $f(x, y, z) = x^{-1}yz$, and let C be the curve parametrized by $\mathbf{r}(t) = (\ln t, t, t^2)$ for $2 \le t \le 4$. Use a computer algebra system to calculate $\int_C f(x, y, z)\, ds$ to four decimal places.

34. CAS Use a CAS to calculate $\int_C \langle e^{x-y}, e^{x+y} \rangle \cdot d\mathbf{r}$ to four decimal places, where C is the curve $y = \sin x$ for $0 \le x \le \pi$, oriented from left to right.

In Exercises 35 and 36, calculate the line integral of $\mathbf{F}(x, y, z) = \langle e^z, e^{x-y}, e^y \rangle$ *over the given path.*

35. The blue path from P to Q in Figure 14

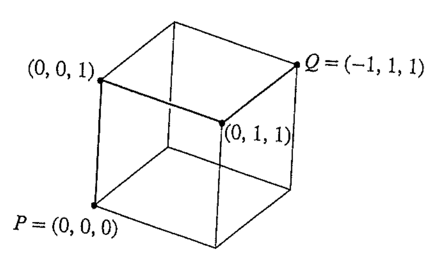


FIGURE 14

The closed path $ABCA$ in Figure 15

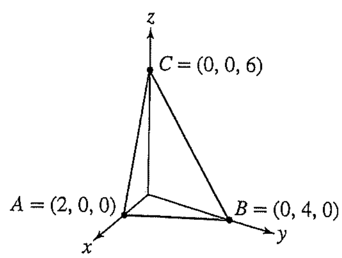


FIGURE 15

In Exercises 37 and 38, C is the path from P to Q in Figure 16 that traces C_1, C_2, and C_3 in the orientation indicated, and $\mathbf{F}$ is a vector field such that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 5, \qquad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 8, \qquad \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 8$$

37. Determine:

(a) $\int_{-C_3} \mathbf{F} \cdot d\mathbf{r}$ **(b)** $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ **(c)** $\int_{-C_1-C_3} \mathbf{F} \cdot d\mathbf{r}$

38. Find the value of $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$, where C' is the path that traverses the loop C_2 four times in the clockwise direction.

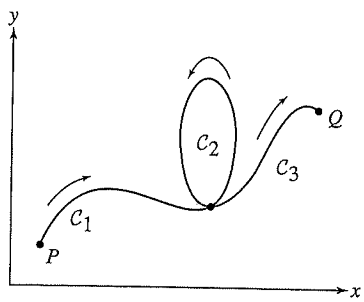


FIGURE 16

39. The values of a function $f(x, y, z)$ and vector field $\mathbf{F}(x, y, z)$ are given at six sample points along the path ABC in Figure 17. Estimate the line integrals of f and $\mathbf{F}$ along ABC.

Point	$f(x, y, z)$	$\mathbf{F}(x, y, z)$
$\left(1, \frac{1}{6}, 0\right)$	3	$\langle 1, 0, 2\rangle$
$\left(1, \frac{1}{2}, 0\right)$	3.3	$\langle 1, 1, 3\rangle$
$\left(1, \frac{5}{6}, 0\right)$	3.6	$\langle 2, 1, 5\rangle$
$\left(1, 1, \frac{1}{6}\right)$	4.2	$\langle 3, 2, 4\rangle$
$\left(1, 1, \frac{1}{2}\right)$	4.5	$\langle 3, 3, 3\rangle$
$\left(1, 1, \frac{5}{6}\right)$	4.2	$\langle 5, 3, 3\rangle$

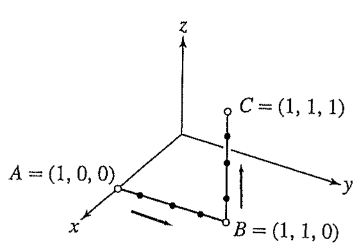


FIGURE 17

40. Estimate the line integrals of $f(x, y)$ and $\mathbf{F}(x, y)$ along the quarter circle (oriented counterclockwise) in Figure 18 using the values at the three sample points along each path.

Point	$f(x, y)$	$\mathbf{F}(x, y)$
A	1	$\langle 1, 2\rangle$
B	-2	$\langle 1, 3\rangle$
C	4	$\langle -2, 4\rangle$

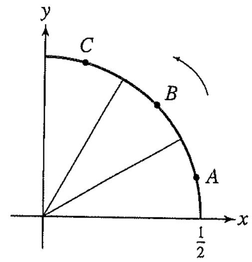


FIGURE 18

41. Determine whether the line integrals of the vector fields around the circle (oriented counterclockwise) in Figure 19 are positive, negative, or zero.

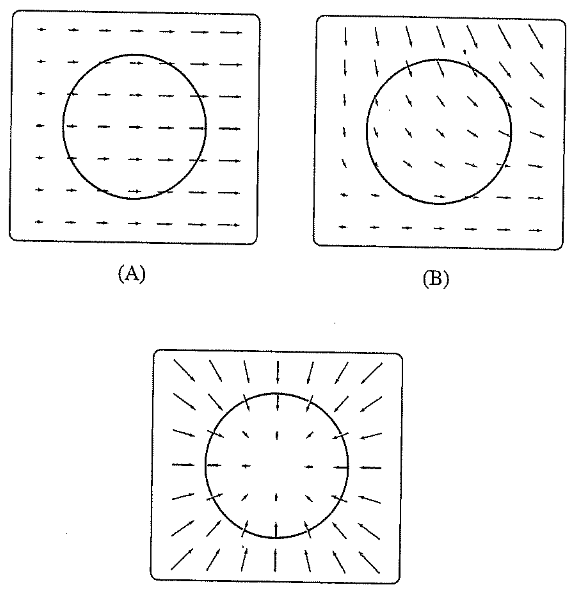


FIGURE 19

42. Determine whether the line integrals of the vector fields along the oriented curves in Figure 20 are positive or negative.

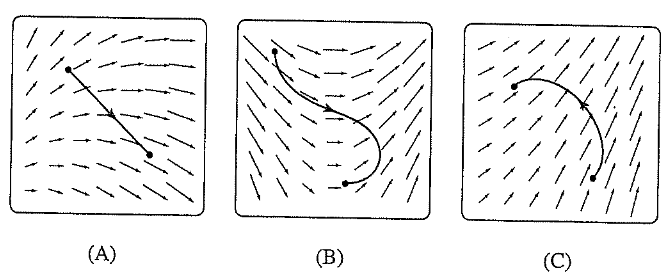


FIGURE 20

43. Calculate the total mass of a circular piece of wire of radius 4 cm centered at the origin whose mass density is $\delta(x, y) = x^2$ g/cm.

44. Calculate the total mass of a metal tube in the helical shape $\mathbf{r}(t) = (\cos t, \sin t, t^2)$ (distance in centimeters) for $0 \le t \le 2\pi$ if the mass density is $\delta(x, y, z) = \sqrt{z}$ g/cm.

45. Find the total charge on the curve $y = x^{4/3}$ for $1 \le x \le 8$ (in centimeters) assuming a charge density of $\delta(x, y) = x/y$ (in units of 10^{-6} C/cm).

46. Find the total charge on the curve $\mathbf{r}(t) = (\sin t, \cos t, \sin^2 t)$ in centimeters for $0 \le t \le \frac{\pi}{8}$ assuming a charge density of $\delta(x, y, z) = xy(y^2 - z)$ (in units of 10^{-6} C/cm).

In Exercises 47–50, use Eq. (6) to compute the electric potential $V(P)$ at the point P for the given charge density (in units of 10^{-6} C).

47. Calculate $V(P)$ at $P = (0, 0, 12)$ if the electric charge is distributed along the quarter circle of radius 4 centered at the origin with charge density $\delta(x, y, z) = xy$.

48. Calculate $V(P)$ at the origin $P = (0, 0)$ if the negative charge is distributed along $y = x^2$ for $1 \le x \le 2$ with charge density $\delta(x, y) = -y\sqrt{x^2 + 1}$.

49. Calculate $V(P)$ at $P = (2, 0, 2)$ if the negative charge is distributed along the y-axis for $1 \le y \le 3$ with charge density $\delta(x, y, z) = -y$.

50. Calculate $V(P)$ at the origin $P = (0, 0)$ if the electric charge is distributed along $y = x^{-1}$ for $\frac{1}{2} \le x \le 2$ with charge density $\delta(x, y) = x^3 y$.

51. Calculate the work done by a field $\mathbf{F} = \langle x + y, x - y \rangle$ when an object moves from $(0, 0)$ to $(1, 1)$ along each of the paths $y = x^2$ and $x = y^2$.

In Exercises 52-54, calculate the work done by the field $\mathbf{F}$ *when the object moves along the given path from the initial point to the final point.*

52. $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$, $\mathbf{r} = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 3\pi$.

53. $\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$, $\mathbf{r} = \left\langle t, t^2, t^3 \right\rangle$ for $0 \le t \le 1$.

54. $\mathbf{F}(x, y, z) = \left\langle e^x, e^y, xyz \right\rangle$, $\mathbf{r} = \left\langle t^2, t, t/2 \right\rangle$ for $0 \le t \le 1$.

55. Figure 21 shows a force field $\mathbf{F}$.

(a) Over which of the two paths, ADC or ABC, does $\mathbf{F}$ perform less work?

(b) If you have to work against $\mathbf{F}$ to move an object from C to A, which of the paths, CBA or CDA, requires less work?

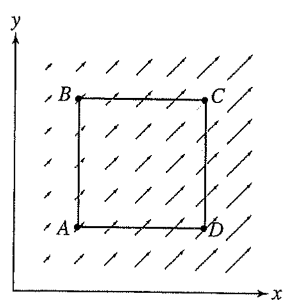


FIGURE 21

56. Verify that the work performed along the segment $\overline{PQ}$ by the constant vector field $\mathbf{F} = \langle 2, -1, 4 \rangle$ is equal to $\mathbf{F} \cdot \overrightarrow{PQ}$ in these cases:

(a) $P = (0, 0, 0)$, $Q = (4, 3, 5)$

(b) $P = (3, 2, 3)$, $Q = (4, 8, 12)$

57. Show that work performed by a constant force field $\mathbf{F}$ ov path C from P to Q is equal to $\mathbf{F} \cdot \overrightarrow{PQ}$.

58. Note that a curve C in polar form $r = f(\theta)$ is parametrized $\mathbf{r}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta))$ because the x- and y-coordinates given by $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Show that $\|\mathbf{r}'(\theta)\| = \sqrt{f(\theta)^2 + f'(\theta)^2}$.

(b) Evaluate $\int_C (x - y)^2\, ds$, where C is the semicircle in Figure 22 with polar equation $r = 2\cos\theta$, $0 \le \theta \le \frac{\pi}{2}$.

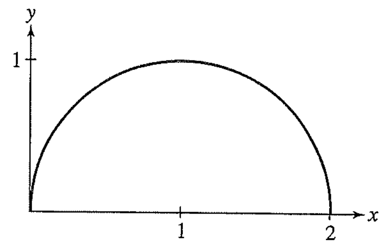


FIGURE 22 Semicircle $r = 2\cos\theta$.

59. Charge is distributed along the spiral with polar equation $r = \theta$ for $0 \le \theta \le 2\pi$. The charge density is $\delta(r, \theta) = r$ (assume distance is in centimeters and charge in units of 10^{-6} C/cm). Use the result of Exercise 58(a) to compute the total charge.

In Exercises 60–63, let $\mathbf{F}$ *be the* **vortex field** *(so-called because it swirls around the origin as in Figure 23):*

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

60. Calculate $I = \int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle of radius 2 centered at the origin. Verify that I changes sign when C is oriented in the clockwise direction.

61. Show that the value of $\int_{C_R} \mathbf{F} \cdot d\mathbf{r}$, where C_R is the circle of radius R centered at the origin and oriented counterclockwise, does not depend on R.

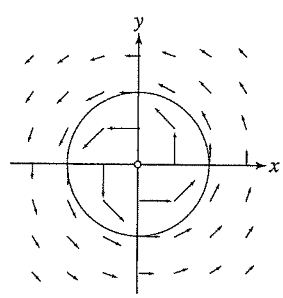


FIGURE 23

62. Let $a > 0$, $b < c$. Show that the integral of $\mathbf{F}$ along the segment [Figure 24(A)] from $P = (a, b)$ to $Q = (a, c)$ is equal to the angle $\angle POQ$.

63. Let C be a curve in polar form $r = f(\theta)$ for $\theta_1 \le \theta \le \theta_2$ [Figure 24(B)], parametrized by $\mathbf{r}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta))$ as in Exercise 58.

(a) Show that the vortex field in polar coordinates is written $\mathbf{F}(r, \theta) = r^{-1}\langle -\sin\theta, \cos\theta \rangle$.

ercises

1. Let $f(x, y, z) = xy\sin(yz)$ and $\mathbf{F} = \nabla f$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is any path from $(0, 0, 0)$ to $(1, 1, \pi)$.

2. Let $\mathbf{F}(x, y, z) = \langle x^{-1}z, y^{-1}z, \ln(xy)\rangle$.
(a) Verify that $\mathbf{F} = \nabla f$, where $f(x, y, z) = z\ln(xy)$.
(b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{r}(t) = \langle e^t, e^{2t}, t^2\rangle$ for $1 \le t \le 3$.
(c) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for any path C from $P = (\frac{1}{2}, 4, 2)$ to $Q = (2, 2, 3)$ contained in the region $x > 0$, $y > 0$.
(d) Why is it necessary to specify that the path lies in the region where x and y are positive?

In Exercises 3–6, verify that $\mathbf{F} = \nabla f$ *and evaluate the line integral of* $\mathbf{F}$ *over the given path.*

3. $\mathbf{F}(x, y) = \langle 3, 6y\rangle$, $f(x, y) = 3x + 3y^2$; $\mathbf{r}(t) = \langle t, 2t^{-1}\rangle$ for $1 \le t \le 4$

4. $\mathbf{F}(x, y) = \langle \cos y, -x\sin y\rangle$, $f(x, y) = x\cos y$; upper half of the unit circle centered at the origin, oriented counterclockwise

5. $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$, $f(x, y, z) = xye^z$; $\mathbf{r}(t) = (t^2, t^3, t - 1)$ for $1 \le t \le 2$

6. $\mathbf{F}(x, y, z) = \frac{z}{x}\mathbf{i} + \mathbf{j} + \ln x\mathbf{k}$, $f(x, y, z) = y + z\ln x$; circle $(x-4)^2 + y^2 = 1$ in the clockwise direction

In Exercises 7–16, find a potential function for $\mathbf{F}$ *or determine that* $\mathbf{F}$ *is not conservative.*

7. $\mathbf{F} = \langle z, 1, x\rangle$

8. $\mathbf{F} = x\mathbf{j} + y\mathbf{k}$

9. $\mathbf{F} = y^2\mathbf{i} + (2xy + e^z)\mathbf{j} + ye^z\mathbf{k}$

10. $\mathbf{F} = \langle y, x, z^3\rangle$

11. $\mathbf{F} = \langle \cos(xz), \sin(yz), xy\sin z\rangle$

12. $\mathbf{F} = \langle \cos z, 2y, -x\sin z\rangle$

13. $\mathbf{F} = \langle z\sec^2 x, z, y + \tan x\rangle$

14. $\mathbf{F} = \langle e^x(z+1), -\cos y, e^x\rangle$

15. $\mathbf{F} = \langle 2xy + 5, x^2 - 4z, -4y\rangle$

16. $\mathbf{F} = \langle yze^{xy}, xze^{xy} - z, e^{xy} - y\rangle$

17. Evaluate

$$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

over the path $\mathbf{r}(t) = (t^2, \sin(\pi t/4), e^{t^2-2t})$ for $0 \le t \le 2$.

18. Evaluate

$$\oint_C \sin x\,dx + z\cos y\,dy + \sin y\,dz$$

where C is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.

In Exercises 19–20, let $\mathbf{F} = \nabla f$, *and determine directly* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *for each of the two paths given, showing that they both give the same answer, which is* $f(Q) - f(P)$.

19. $f = x^2y - z$, $\mathbf{r}_1 = \langle t, t, 0\rangle$ for $0 \le t \le 1$, and $\mathbf{r}_2 = \langle t, t^2, 0\rangle$ for $0 \le t \le 1$

20. $f = zy + xy + xz$, $\mathbf{r}_1 = \langle t, t, t\rangle$ for $0 \le t \le 1$, $\mathbf{r}_2 = \langle t, t^2, t^3\rangle$ for $0 \le t \le 1$

21. A vector field $\mathbf{F}$ and contour lines of a potential function for $\mathbf{F}$ are shown in Figure 17. Calculate the common value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the curves shown in Figure 17 oriented in the direction from P to Q.

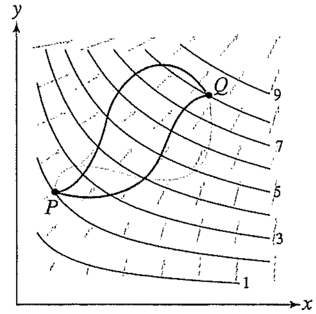


FIGURE 17

22. Give a reason why the vector field $\mathbf{F}$ in Figure 18 is not conservative.

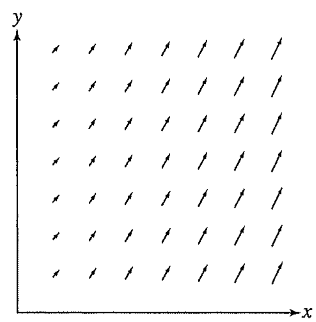


FIGURE 18

23. Calculate the work expended when a particle is moved from O to Q along segments $\overline{OP}$ and $\overline{PQ}$ in Figure 19 in the presence of the force field $\mathbf{F} = \langle x^2, y^2\rangle$. How much work is expended moving in a complete circuit around the square?

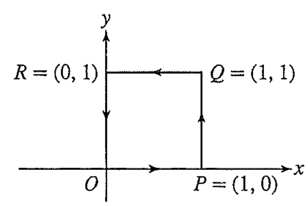


FIGURE 19

24. Let $\mathbf{F}(x, y) = \left\langle \frac{1}{x}, \frac{-1}{y}\right\rangle$. Calculate the work against F required to move an object from $(1, 1)$ to $(3, 4)$ along any path in the first quadrant.

25. Compute the work W against the earth's gravitational field required to move a satellite of mass $m = 1000$ kg along any path from an orbit of altitude 4000 km to an orbit of altitude 6000 km.

26. An electric dipole with dipole moment $p = 4 \times 10^{-5}$ C-m sets up an electric field (in newtons per coulomb)

$$\mathbf{F}(x, y, z) = \frac{kp}{r^5}\left\langle 3xz, 3yz, 2z^2 - x^2 - y^2 \right\rangle$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ with distance in meters and $k = 8.99 \times 10^9$ N-m^2/C^2. Calculate the work against $\mathbf{F}$ required to move a particle of charge $q = 0.01$ C from $(1, -5, 0)$ to $(3, 4, 4)$. *Note:* The force on q is $q\mathbf{F}$ newtons.

27. On the surface of the earth, the gravitational field (with z as vertical coordinate measured in meters) is $\mathbf{F} = \langle 0, 0, -g \rangle$.

(a) Find a potential function for $\mathbf{F}$.

(b) Beginning at rest, a ball of mass $m = 2$ kg moves under the influence of gravity (without friction) along a path from $P = (3, 2, 400)$ to $Q = (-21, 40, 50)$. Find the ball's velocity when it reaches Q.

28. An electron at rest at $P = (5, 3, 7)$ moves along a path ending at $Q = (1, 1, 1)$ under the influence of the electric field (in newtons per coulomb)

$$\mathbf{F}(x, y, z) = 400(x^2 + z^2)^{-1} \langle x, 0, z \rangle$$

(a) Find a potential function for $\mathbf{F}$.

(b) What is the electron's speed at point Q? Use Conservation of Energy and the value $q_e/m_e = -1.76 \times 10^{11}$ C/kg, where q_e and m_e are the charge and mass on the electron, respectively.

29. Let $\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$ be the vortex field. Determine $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each of the paths in Figure 20.

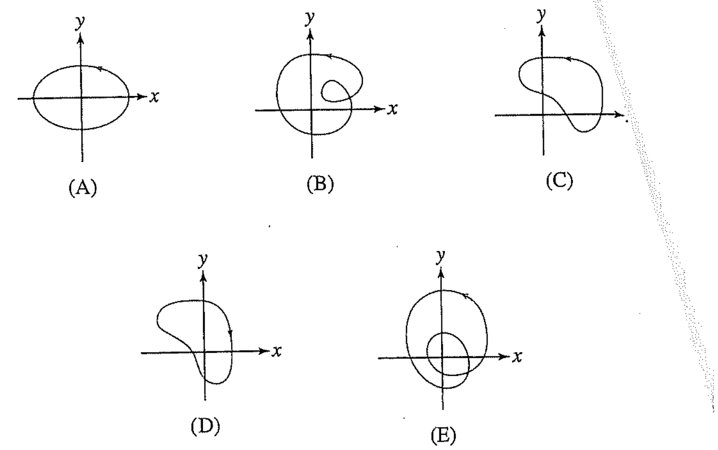


FIGURE 20

30. The vector field $\mathbf{F}(x, y) = \left\langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right\rangle$ is defined on the domain $\mathcal{D} = \{(x, y) \neq (0, 0)\}$.

(a) Is $\mathcal{D}$ simply connected?

(b) Show that $\mathbf{F}$ satisfies the cross-partial condition. Does this guarantee that $\mathbf{F}$ is conservative?

(c) Show that $\mathbf{F}$ is conservative on $\mathcal{D}$ by finding a potential function.

(d) Do these results contradict Theorem 4?

Further Insights and Challenges

31. Suppose that $\mathbf{F}$ is defined on $\mathbf{R}^3$ and that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths C in $\mathbf{R}^3$. Prove:

(a) $\mathbf{F}$ is path-independent; that is, for any two paths C_1 and C_2 in $\mathcal{D}$ with the same initial and terminal points,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

(b) $\mathbf{F}$ is conservative.

16.4 Parametrized Surfaces and Surface Integrals

The basic idea of an integral appears in several guises. So far, we have defined single, double, and triple integrals and, in the previous section, line integrals over curves. Now we consider one last type on integral: integrals over surfaces. We treat scalar surface integrals in this section and vector surface integrals in the following section.

Just as parametrized curves are a key ingredient in the discussion of line integrals, surface integrals require the notion of a **parametrized surface**—that is, a surface $\mathcal{S}$ whose points are described in the form

$$G(u, v) = (x(u, v), y(u, v), z(u, v))$$

The variables u, v (called parameters) vary in a region $\mathcal{D}$ called the **parameter domain.** Two parameters u and v are needed to parametrize a surface because the surface is two-dimensional.

Figure 1 shows a plot of the surface $\mathcal{S}$ with the parametrization

$$G(u, v) = (u + v, u^3 - v, v^3 - u)$$

This surface consists of all points (x, y, z) in $\mathbf{R}^3$ such that

$$x = u + v, \qquad y = u^3 - v, \qquad z = v^3 - u$$

for (u, v) in $\mathcal{D} = \mathbf{R}^2$.

– Graph of $z = g(x, y)$:

$$G(x, y) = (x, y, g(x, y))$$
$$\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_y = \langle -g_x, -g_y, 1 \rangle$$
$$dS = \|\mathbf{N}\|\, dx\, dy = \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy$$

16.4 EXERCISES

Preliminary Questions

1. What is the surface integral of the function $f(x, y, z) = 10$ over a surface of total area 5?

2. What interpretation can we give to the length $\|\mathbf{N}\|$ of the normal vector for a parametrization $G(u, v)$?

3. A parametrization maps a rectangle of size 0.01×0.02 in the uv-plane onto a small patch $\mathcal{S}$ of a surface. Estimate Area($\mathcal{S}$) if $\mathbf{T}_u \times \mathbf{T}_v = \langle 1, 2, 2 \rangle$ at a sample point in the rectangle.

4. A small surface $\mathcal{S}$ is divided into three small pieces, each of area 0.2. Estimate $\iint_{\mathcal{S}} f(x, y, z)\, dS$ if $f(x, y, z)$ takes the values 0.9, 1, and 1.1 at sample points in these three pieces.

5. A surface $\mathcal{S}$ has a parametrization whose domain is the square $0 \le u, v \le 2$ such that $\|\mathbf{N}(u, v)\| = 5$ for all (u, v). What is Area($\mathcal{S}$)?

6. What is the outward-pointing unit normal to the sphere of radius 3 centered at the origin at $P = (2, 2, 1)$?

Exercises

1. Match each parametrization with the corresponding surface in Figure 16.

(a) $(u, \cos v, \sin v)$
(b) $(u, u + v, v)$
(c) (u, v^3, v)
(d) $(\cos u \sin v, 3 \cos u \sin v, \cos v)$
(e) $(u, u(2 + \cos v), u(2 + \sin v))$

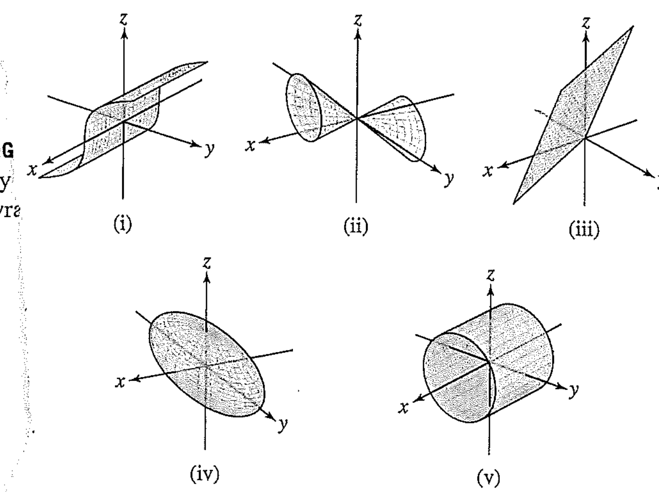


FIGURE 16

2. Show that $G(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$ parametrizes the paraboloid $z = 1 - x^2 - y^2$. Describe the grid curves of this parametrization.

3. Show that $G(u, v) = (2u + 1, u - v, 3u + v)$ parametrizes the plane $2x - y - z = 2$. Then:

(a) Calculate $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{N}(u, v)$.
(b) Find the area of $\mathcal{S} = G(\mathcal{D})$, where $\mathcal{D} = \{(u, v) : 0 \le u \le 2, 0 \le v \le 1\}$.
(c) Express $f(x, y, z) = yz$ in terms of u and v, and evaluate $\iint_{\mathcal{S}} f(x, y, z)\, dS$.

4. Let $\mathcal{S} = G(\mathcal{D})$, where $\mathcal{D} = \{(u, v) : u^2 + v^2 \le 1, u \ge 0, v \ge 0\}$ and G is as defined in Exercise 3.

(a) Calculate the surface area of $\mathcal{S}$.
(b) Evaluate $\iint_{\mathcal{S}} (x - y)\, dS$. *Hint:* Use polar coordinates.

5. Let $G(x, y) = (x, y, xy)$.

(a) Calculate $\mathbf{T}_x$, $\mathbf{T}_y$, and $\mathbf{N}(x, y)$.
(b) Let $\mathcal{S}$ be the part of the surface with parameter domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$. Verify the following formula and evaluate using polar coordinates:

$$\iint_{\mathcal{S}} 1\, dS = \iint_{\mathcal{D}} \sqrt{1 + x^2 + y^2}\, dx\, dy$$

(c) Verify the following formula and evaluate:

$$\iint_{\mathcal{S}} z\, dS = \int_0^{\pi/2} \int_0^1 (\sin\theta \cos\theta) r^3 \sqrt{1 + r^2}\, dr\, d\theta$$

6. A surface $\mathcal{S}$ has a parametrization $G(u, v)$ whose domain $\mathcal{D}$ is the square in Figure 17. Suppose that G has the following normal vectors:

$$\mathbf{N}(A) = \langle 2, 1, 0 \rangle, \quad \mathbf{N}(B) = \langle 1, 3, 0 \rangle$$
$$\mathbf{N}(C) = \langle 3, 0, 1 \rangle, \quad \mathbf{N}(D) = \langle 2, 0, 1 \rangle$$

Estimate $\iint_{\mathcal{S}} f(x, y, z)\, dS$, where f is a function such that $f(G(u, v)) = u + v$.

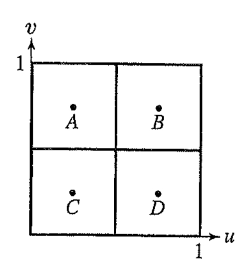


FIGURE 17

In Exercises 7–10, calculate $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{N}(u, v)$ for the parametrized surface at the given point. Then find the equation of the tangent plane to the surface at that point.

7. $G(u, v) = (2u + v, u - 4v, 3u)$; $\quad u = 1, \quad v = 4$

8. $G(u, v) = (u^2 - v^2, u + v, u - v)$; $\quad u = 2, \quad v = 3$

9. $G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$; $\quad \theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{4}$

10. $G(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$; $\quad r = \frac{1}{2}, \quad \theta = \frac{\pi}{4}$

11. Use the normal vector computed in Exercise 8 to estimate the area of the small patch of the surface $G(u, v) = (u^2 - v^2, u + v, u - v)$ defined by

$$2 \le u \le 2.1, \qquad 3 \le v \le 3.2$$

12. Sketch the small patch of the sphere whose spherical coordinates satisfy

$$\frac{\pi}{2} - 0.15 \le \theta \le \frac{\pi}{2} + 0.15, \qquad \frac{\pi}{4} - 0.1 \le \phi \le \frac{\pi}{4} + 0.1$$

Use the normal vector computed in Exercise 9 to estimate its area.

In Exercises 13–26, calculate $\iint_S f(x, y, z)\, dS$ for the given surface and function.

13. $G(u, v) = (u\cos v, u\sin v, u)$, $\ 0 \le u \le 1$, $\quad 0 \le v \le 1$;
$f(x, y, z) = z(x^2 + y^2)$

14. $G(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$, $\ 0 \le r \le 1$, $\ 0 \le \theta \le 2\pi$;
$f(x, y, z) = \sqrt{x^2 + y^2}$

15. $y = 9 - z^2$, $\ 0 \le x \le 3, 0 \le z \le 3$; $\quad f(x, y, z) = z$

16. $y = 9 - z^2$, $\ 0 \le x \le z \le 3$; $\quad f(x, y, z) = 1$

17. $x^2 + y^2 + z^2 = 1$, $\ x, y, z \ge 0$; $\ f(x, y, z) = x^2$.

18. $z = 4 - x^2 - y^2$, $\ 0 \le z \le 3$; $\quad f(x, y, z) = x^2/(4 - z)$

19. $x^2 + y^2 = 4$, $\ 0 \le z \le 4$; $\quad f(x, y, z) = e^{-z}$

20. $G(u, v) = (u, v^3, u + v)$, $\ 0 \le u \le 1, 0 \le v \le 1$;
$f(x, y, z) = y$

21. Part of the plane $x + y + z = 1$, where $x, y, z \ge 0$;
$f(x, y, z) = z$

22. Part of the plane $x + y + z = 0$ contained in the cylinder $x^2 + y^2 = 1$; $\quad f(x, y, z) = z^2$

23. $x^2 + y^2 + z^2 = 4, 1 \le z \le 2$;
$f(x, y, z) = z^2(x^2 + y^2 + z^2)^{-1}$

24. $x^2 + y^2 + z^2 = 4, 0 \le y \le 1$; $\ f(x, y, z) = y$

25. Part of the surface $z = x^3$, where $0 \le x \le 1$, $0 \le y \le 1$;
$f(x, y, z) = z$

26. Part of the unit sphere centered at the origin, where $x \ge 0$ and $|y| \le x$; $\quad f(x, y, z) = x$

27. A surface S has a parametrization $G(u, v)$ with domain $0 \le u \le 2$, $0 \le v \le 4$ such that the following partial derivatives are constant:

$$\frac{\partial G}{\partial u} = \langle 2, 0, 1 \rangle, \qquad \frac{\partial G}{\partial v} = \langle 4, 0, 3 \rangle$$

What is the surface area of S?

28. Let S be the sphere of radius R centered at the origin. Explain symmetry:

$$\iint_S x^2\, dS = \iint_S y^2\, dS = \iint_S z^2\, dS$$

Then show that $\iint_S x^2\, dS = \frac{4}{3}\pi R^4$ by adding the integrals.

29. Calculate $\iint_S (xy + e^z)\, dS$, where S is the triangle in Figure 18 with vertices $(0, 0, 3)$, $(1, 0, 2)$, and $(0, 4, 1)$.

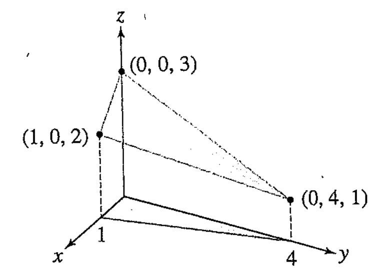


FIGURE 18

30. Use spherical coordinates to compute the surface area of a sphere of radius R.

31. Use cylindrical coordinates to compute the surface area of a sphere of radius R.

32. CAS Let S be the surface with parametrization

$$G(u, v) = \big((3 + \sin v)\cos u,\ (3 + \sin v)\sin u,\ v\big)$$

for $0 \le u \le 2\pi, 0 \le v \le 2\pi$. Using a computer algebra system:
(a) plot S from several different viewpoints. Is S best described as a "vase that holds water" or a "bottomless vase"?
(b) calculate the normal vector $\mathbf{N}(u, v)$.
(c) calculate the surface area of S to four decimal places.

33. CAS Let S be the surface $z = \ln(5 - x^2 - y^2)$ for $0 \le x \le 1$, $0 \le y \le 1$. Using a computer algebra system:
(a) calculate the surface area of S to four decimal places.
(b) calculate $\iint_S x^2 y^3\, dS$ to four decimal places.

34. Find the area of the portion of the plane $2x + 3y + 4z = 28$ lying above the rectangle $1 \le x \le 3$, $2 \le y \le 5$ in the xy-plane.

35. What is the area of the portion of the plane $2x + 3y + 4z = 28$ lying above the domain $\mathcal{D}$ in the xy-plane in Figure 19 if Area($\mathcal{D}$) = 5

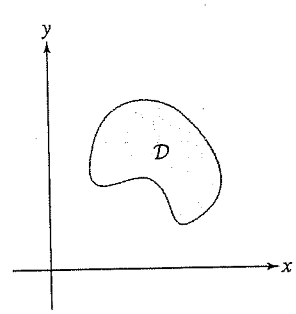


FIGURE 19

36. Find the surface area of the part of the cone $x^2 + y^2 = z^2$ between the planes $z = 2$ and $z = 5$.

4. If $\mathbf{F}(P) = \mathbf{n}(P)$ at each point on $\mathcal{S}$, then $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ is equal to which of the following?
(a) Zero (b) Area($\mathcal{S}$) (c) Neither

5. Let $\mathcal{S}$ be the disk $x^2 + y^2 \leq 1$ in the xy-plane oriented with normal in the positive z-direction. Determine $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ for each of the following vector constant fields:
(a) $\mathbf{F} = \langle 1, 0, 0 \rangle$ (b) $\mathbf{F} = \langle 0, 0, 1 \rangle$ (c) $\mathbf{F} = \langle 1, 1, 1 \rangle$

6. Estimate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathcal{S}$ is a tiny oriented surface of area 0.05 and the value of $\mathbf{F}$ at a sample point in $\mathcal{S}$ is a vector of length 2 making an angle $\frac{\pi}{4}$ with the normal to the surface.

7. A small surface $\mathcal{S}$ is divided into three pieces of area 0.2. Estimate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F}$ is a unit vector field making angles of 85°, 90°, and 95° with the normal at sample points in these three pieces.

Exercises

1. Let $\mathbf{F} = \langle z, 0, y \rangle$, and let $\mathcal{S}$ be the oriented surface parametrized by $G(u, v) = (u^2 - v, u, v^2)$ for $0 \leq u \leq 2$, $-1 \leq v \leq 4$. Calculate:
(a) $\mathbf{N}$ and $\mathbf{F} \cdot \mathbf{N}$ as functions of u and v
(b) The normal component of $\mathbf{F}$ to the surface at $P = (3, 2, 1) = G(2, 1)$
(c) $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$

2. Let $\mathbf{F} = \langle y, -x, x^2 + y^2 \rangle$, and let $\mathcal{S}$ be the portion of the paraboloid $z = x^2 + y^2$ where $x^2 + y^2 \leq 3$.
(a) Show that if $\mathcal{S}$ is parametrized in polar variables $x = r\cos\theta$, $y = r\sin\theta$, then $\mathbf{F} \cdot \mathbf{N} = r^3$.
(b) Show that $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{3} r^3 \, dr \, d\theta$ and evaluate.

3. Let $\mathcal{S}$ be the unit square in the xy-plane shown in Figure 14, oriented with the normal pointing in the positive z-direction. Estimate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}$ is a vector field whose values at the labeled points are

$$\mathbf{F}(A) = \langle 2, 6, 4 \rangle, \qquad \mathbf{F}(B) = \langle 1, 1, 7 \rangle$$
$$\mathbf{F}(C) = \langle 3, 3, -3 \rangle, \qquad \mathbf{F}(D) = \langle 0, 1, 8 \rangle$$

4. Suppose that $\mathcal{S}$ is a surface in $\mathbf{R}^3$ with a parametrization G whose domain $\mathcal{D}$ is the square in Figure 14. The values of a function f, a vector field $\mathbf{F}$, and the normal vector $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$ at $G(P)$ are given for the four sample points in $\mathcal{D}$ in the following table. Estimate the surface integrals of f and $\mathbf{F}$ over $\mathcal{S}$.

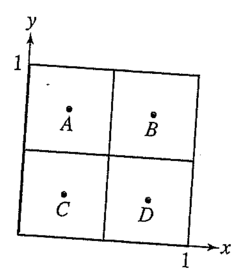


FIGURE 14

Point P in $\mathcal{D}$	f	$\mathbf{F}$	$\mathbf{N}$
A	3	$\langle 2, 6, 4 \rangle$	$\langle 1, 1, 1 \rangle$
B	1	$\langle 1, 1, 7 \rangle$	$\langle 1, 1, 0 \rangle$
C	2	$\langle 3, 3, -3 \rangle$	$\langle 1, 0, -1 \rangle$
D	5	$\langle 0, 1, 8 \rangle$	$\langle 2, 1, 0 \rangle$

In Exercises 5–17, compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ *for the given oriented surface.*

5. $\mathbf{F} = \langle y, z, x \rangle$, plane $3x - 4y + z = 1$, $0 \leq x \leq 1$, $0 \leq y \leq 1$, upward-pointing normal

6. $\mathbf{F} = \langle e^z, z, x \rangle$, $G(r, s) = (rs, r + s, r)$, $0 \leq r \leq 1$, $0 \leq s \leq 1$, oriented by $\mathbf{T}_r \times \mathbf{T}_s$

7. $\mathbf{F} = \langle 0, 3, x \rangle$, part of sphere $x^2 + y^2 + z^2 = 9$, where $x \geq 0$, $y \geq 0$, $z \geq 0$, outward-pointing normal

8. $\mathbf{F} = \langle x, y, z \rangle$, part of sphere $x^2 + y^2 + z^2 = 1$, where $\frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}$, inward-pointing normal

9. $\mathbf{F} = \langle z, z, x \rangle$, $z = 9 - x^2 - y^2$, $x \geq 0$, $y \geq 0$, $z \geq 0$, upward-pointing normal

10. $\mathbf{F} = \langle \sin y, \sin z, yz \rangle$, rectangle $0 \leq y \leq 2$, $0 \leq z \leq 3$ in the (y, z)-plane, normal pointing in negative x-direction

11. $\mathbf{F} = y^2\mathbf{i} + 2\mathbf{j} - x\mathbf{k}$, portion of the plane $x + y + z = 1$ in the octant $x, y, z \geq 0$, upward-pointing normal

12. $\mathbf{F} = \langle x, y, e^z \rangle$, cylinder $x^2 + y^2 = 4$, $1 \leq z \leq 5$, outward-pointing normal

13. $\mathbf{F} = \langle xz, yz, z^{-1} \rangle$, disk of radius 3 at height 4 parallel to the xy-plane, upward-pointing normal

14. $\mathbf{F} = \langle xy, y, 0 \rangle$, cone $z^2 = x^2 + y^2$, $x^2 + y^2 \leq 4$, $z \geq 0$, downward-pointing normal

15. $\mathbf{F} = \langle 0, 0, e^{y+z} \rangle$, boundary of unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, outward-pointing normal

16. $\mathbf{F} = \langle 0, 0, z^2 \rangle$, $G(u, v) = (u\cos v, u\sin v, v)$, $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$, upward-pointing normal

17. $\mathbf{F} = \langle y, z, 0 \rangle$, $G(u, v) = (u^3 - v, u + v, v^2)$, $0 \leq u \leq 2$, $0 \leq v \leq 3$, downward-pointing normal

18. Let $\mathcal{S}$ be the oriented half-cylinder in Figure 15. In (a)–(f), determine whether $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ is positive, negative, or zero. Explain your reasoning.
(a) $\mathbf{F} = \mathbf{i}$ (b) $\mathbf{F} = \mathbf{j}$ (c) $\mathbf{F} = \mathbf{k}$
(d) $\mathbf{F} = y\mathbf{i}$ (e) $\mathbf{F} = -y\mathbf{j}$ (f) $\mathbf{F} = x\mathbf{j}$

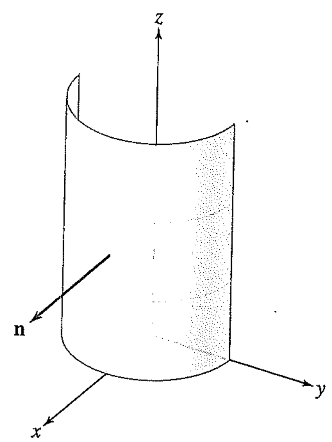


FIGURE 15

19. Let $\mathbf{e_r} = \langle x/r, y/r, z/r \rangle$ be the unit radial vector, where $r = \sqrt{x^2 + y^2 + z^2}$. Calculate the integral of $\mathbf{F} = e^{-r}\mathbf{e_r}$ over:

(a) the upper hemisphere of $x^2 + y^2 + z^2 = 9$, outward-pointing normal.

(b) the octant $x \geq 0, y \geq 0, z \geq 0$ of the unit sphere centered at the origin.

20. Show that the flux of $\mathbf{F} = \dfrac{\mathbf{e}_r}{r^2}$ through a sphere centered at the origin does not depend on the radius of the sphere.

21. The electric field due to a point charge located at the origin in $\mathbf{R}^3$ is $\mathbf{E} = k\dfrac{\mathbf{e}_r}{r^2}$, where $r = \sqrt{x^2 + y^2 + z^2}$ and k is a constant. Calculate the flux of $\mathbf{E}$ through the disk D of radius 2 parallel to the xy-plane with center $(0, 0, 3)$.

22. Let $\mathcal{S}$ be the ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$. Calculate the flux of $\mathbf{F} = z\mathbf{i}$ over the portion of $\mathcal{S}$ where $x, y, z \leq 0$ with upward-pointing normal. *Hint:* Parametrize $\mathcal{S}$ using a modified form of spherical coordinates (θ, ϕ).

23. Let $\mathbf{v} = z\mathbf{k}$ be the velocity field (in meters per second) of a fluid in $\mathbf{R}^3$. Calculate the flow rate (in cubic meters per second) through the upper hemisphere ($z \geq 0$) of the sphere $x^2 + y^2 + z^2 = 1$.

24. Calculate the flow rate of a fluid with velocity field $\mathbf{v} = \langle x, y, x^2y \rangle$ (in meters per second) through the portion of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ in the xy-plane, where $x, y \geq 0$, oriented with the normal in the positive z-direction.

In Exercises 25–28, a net is dipped in a river. Determine the flow rate of water across the net if the velocity vector field for the river is given by $\mathbf{v}$ *and the net is described by the given equations.*

25. $\mathbf{v} = \langle x - y, z + y + 4, z^2 \rangle$, net given by $x^2 + z^2 \leq 1$, $y = 0$, oriented in the positive y-direction

26. $\mathbf{v} = \langle x - y, z + y + 4, z^2 \rangle$, net given by $y = 1 - x^2 - z^2$, $y \geq 0$, oriented in the positive y-direction

27. $\mathbf{v} = \langle x - y, z + y + 4, z^2 \rangle$, net given by $y = \sqrt{1 - x^2 - z^2}$, $y \geq 0$, oriented in the positive y-direction

28. $\mathbf{v} = \langle zy, xz, xy \rangle$, net given by $y = 1 - x - z$, for $x, y, z \geq 0$ oriented in the positive y-direction

In Exercises 29–30, let $\mathcal{T}$ be the triangular region with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented with upward-pointing normal vector (Figure 16). Assume distances are in meters.

29. A fluid flows with constant velocity field $\mathbf{v} = 2\mathbf{k}$ (meters per second or m/s). Calculate:

(a) the flow rate through $\mathcal{T}$.

(b) the flow rate through the projection of $\mathcal{T}$ onto the xy-plane [the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 0)$].

30. Calculate the flow rate through $\mathcal{T}$ if $\mathbf{v} = -\mathbf{j}$ m/s.

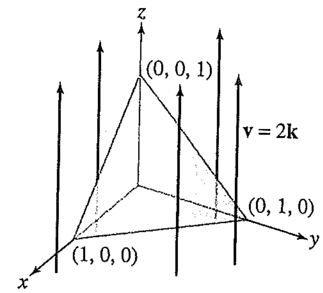


FIGURE 16

31. Prove that if $\mathcal{S}$ is the part of a graph $z = g(x, y)$ lying over a domain $\mathcal{D}$ in the xy-plane, then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \left(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \right) dx\, dy$$

In Exercises 32–33, a varying current $i(t)$ flows through a long straight wire in the xy-plane as in Example 5. The current produces a magnetic field $\mathbf{B}$ *whose magnitude at a distance r from the wire is* $B = \dfrac{\mu_0 i}{2\pi r}$ *T, where* $\mu_0 = 4\pi \cdot 10^{-7}$ *T-m/A. Furthermore,* $\mathbf{B}$ *points into the page at points P in the xy-plane.*

32. Assume that $i(t) = t(12 - t)$ A (t in seconds). Calculate the flux $\Phi(t)$, at time t, of $\mathbf{B}$ through a rectangle of dimensions $L \times H = 3 \times 2$ m whose top and bottom edges are parallel to the wire and whose bottom edge is located $d = 0.5$ m above the wire, similar to Figure 13(B). Then use Faraday's Law to determine the voltage drop around the rectangular loop (the boundary of the rectangle) at time t.

33. Assume that $i = 10e^{-0.1t}$ A (t in seconds). Calculate the flux $\Phi(t)$, at time t, of $\mathbf{B}$ through the isosceles triangle of base 12 cm and height 6 cm whose bottom edge is 3 cm from the wire, as in Figure 17. Assume the triangle is oriented with normal vector pointing out of the page. Use Faraday's Law to determine the voltage drop around the triangular loop (the boundary of the triangle) at time t.

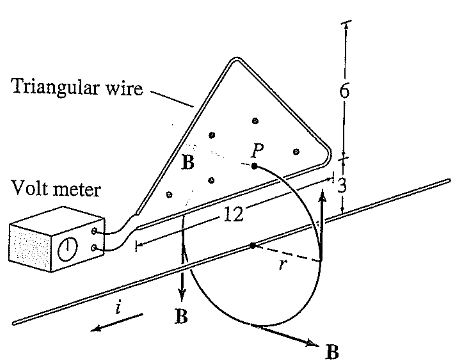


FIGURE 17

- Formulas for the area of the region $\mathcal{D}$ enclosed by $\mathcal{C}$:

$$\text{Area}(\mathcal{D}) = \oint_{\mathcal{C}} x\,dy = \oint_{\mathcal{C}} -y\,dx = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx$$

- The quantity

$$\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

is interpreted as *circulation per unit area.* If $\mathcal{D}$ is a small domain with boundary $\mathcal{C}$, then for any $P \in \mathcal{D}$,

$$\oint_{\mathcal{C}} F_1\,dx + F_2\,dy \approx \text{curl}_z(\mathbf{F})(P) \cdot \text{Area}(\mathcal{D})$$

- Vector Form of Green's Theorem:

$$\oint_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_{\mathcal{D}} \text{div}(\mathbf{F})\,dA$$

17.1 EXERCISES

Preliminary Questions

1. Which vector field $\mathbf{F}$ is being integrated in the line integral $\oint x^2\,dy - e^y\,dx$?

2. Draw a domain in the shape of an ellipse and indicate with an arrow the boundary orientation of the boundary curve. Do the same for the annulus (the region between two concentric circles).

3. The circulation of a conservative vector field around a closed curve is zero. Is this fact consistent with Green's Theorem? Explain.

4. Indicate which of the following vector fields possess this property: For every simple closed curve $\mathcal{C}$, $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is equal to the area enclosed by $\mathcal{C}$.

(a) $\mathbf{F}(x, y) = \langle -y, 0\rangle$

(b) $\mathbf{F}(x, y) = \langle x, y\rangle$

(c) $\mathbf{F}(x, y) = \langle \sin(x^2), x + e^{y^2}\rangle$

Exercises

1. Verify Green's Theorem for the line integral $\oint_{\mathcal{C}} xy\,dx + y\,dy$, where $\mathcal{C}$ is the unit circle, oriented counterclockwise.

2. Let $I = \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y + \sin x^2, x^2 + e^{y^2}\rangle$ and $\mathcal{C}$ is the circle of radius 4 centered at the origin.

(a) Which is easier, evaluating I directly or using Green's Theorem?

(b) Evaluate I using the easier method.

In Exercises 3–10, use Green's Theorem to evaluate the line integral. Orient the curve counterclockwise unless otherwise indicated.

3. $\oint_{\mathcal{C}} y^2\,dx + x^2\,dy$, where $\mathcal{C}$ is the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$

4. $\oint_{\mathcal{C}} e^{2x+y}\,dx + e^{-y}\,dy$, where $\mathcal{C}$ is the triangle with vertices (0, 0), (1, 0), and (1, 1)

5. $\oint_{\mathcal{C}} x^2 y\,dx$, where $\mathcal{C}$ is the unit circle centered at the origin

6. $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x + y, x^2 - y\rangle$ and $\mathcal{C}$ is the boundary of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ for $0 \le x \le 1$

7. $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2, x^2\rangle$ and $\mathcal{C}$ consists of the arcs $y = x^2$ and $y = x$ for $0 \le x \le 1$

8. $\oint_{\mathcal{C}} (\ln x + y)\,dx - x^2\,dy$, where $\mathcal{C}$ is the rectangle with vertices (1, 1), (3, 1), (1, 4), and (3, 4)

9. The line integral of $\mathbf{F}(x, y) = \langle e^{x+y}, e^{x-y}\rangle$ along the curve (oriented clockwise) consisting of the line segments by joining the points (0, 0), (2, 2), (4, 2), (2, 0), and back to (0, 0) (Note the orientation.)

10. $\int_{\mathcal{C}} xy\,dx + (x^2 + x)\,dy$, where $\mathcal{C}$ is the path in Figure 17

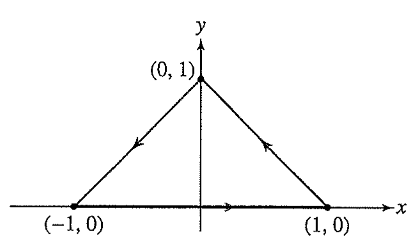


FIGURE 17

11. Let $\mathbf{F}(x, y) = \langle 2xe^y, x + x^2e^y \rangle$ and let C be the quarter-circle path from A to B in Figure 18. Evaluate $I = \oint_C \mathbf{F} \cdot d\mathbf{r}$ as follows:

(a) Find a function $f(x, y)$ such that $\mathbf{F} = \mathbf{G} + \nabla f$, where $\mathbf{G} = \langle 0, x \rangle$.

(b) Show that the line integrals of $\mathbf{G}$ along the segments $\overline{OA}$ and $\overline{OB}$ are zero.

(c) Evaluate I. *Hint:* Use Green's Theorem to show that

$$I = f(B) - f(A) + 4\pi$$

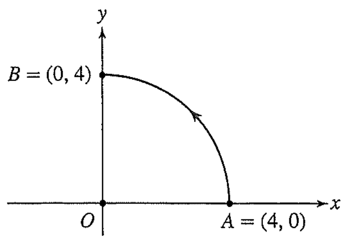


FIGURE 18

12. Compute the line integral of $\mathbf{F}(x, y) = \langle x^3, 4x \rangle$ along the path from A to B in Figure 19. To save work, use Green's Theorem to relate this line integral to the line integral along the vertical path from B to A.

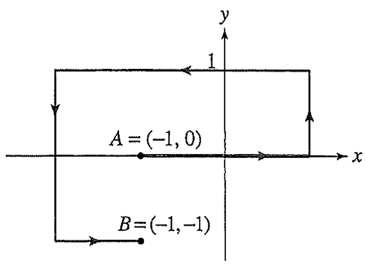


FIGURE 19

13. Evaluate $I = \int_C (\sin x + y)\, dx + (3x + y)\, dy$ for the nonclosed path $ABCD$ in Figure 20. Use the method of Exercise 12.

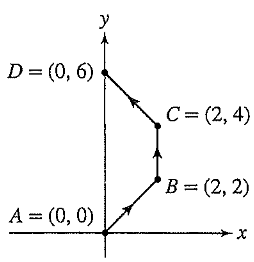


FIGURE 20

In Exercises 14–17, use one of the formulas in Eq. (6) to calculate the area of the given region.

14. The circle of radius 3 centered at the origin

15. The triangle with vertices (0, 0), (1, 0), and (1, 1)

16. The region between the x-axis and the cycloid parametrized by $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ for $0 \le t \le 2\pi$ (Figure 21)

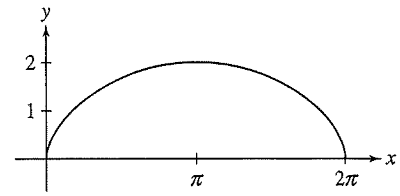


FIGURE 21 Cycloid.

17. The region between the graph of $y = x^2$ and the x-axis for $0 \le x \le 2$

18. A square with vertices (1, 1), (−1, 1), (−1, −1), and (1, −1) has area 4. Calculate this area three times using the formulas in Eq. (6).

19. Let $x^3 + y^3 = 3xy$ be the **folium of Descartes** (Figure 22).

(a) Show that the folium has a parametrization in terms of $t = y/x$ given by

$$x = \frac{3t}{1 + t^3}, \qquad y = \frac{3t^2}{1 + t^3} \qquad (-\infty < t < \infty) \quad (t \neq -1)$$

(b) Show that

$$x\, dy - y\, dx = \frac{9t^2}{(1 + t^3)^2}\, dt$$

Hint: By the Quotient Rule,

$$x^2\, d\left(\frac{y}{x}\right) = x\, dy - y\, dx$$

(c) Find the area of the loop of the folium. *Hint:* The limits of integration are 0 and ∞.

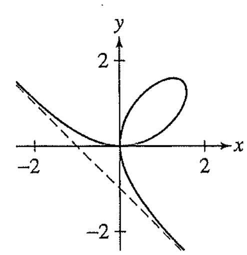


FIGURE 22 Folium of Descartes.

20. Find a parametrization of the lemniscate $(x^2 + y^2)^2 = xy$ (see Figure 23) by using $t = y/x$ as a parameter (see Exercise 19). Then use Eq. (6) to find the area of one loop of the lemniscate.

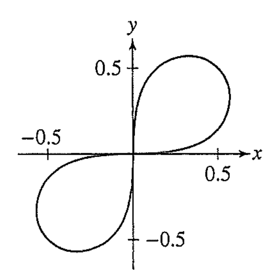


FIGURE 23 Lemniscate.

21. The Centroid via Boundary Measurements The centroid (see Section 15.5) of a domain $\mathcal{D}$ enclosed by a simple closed curve C is the point with coordinates $(\overline{x}, \overline{y}) = (M_y/M, M_x/M)$, where M is the area of $\mathcal{D}$ and the moments are defined by

$$M_x = \iint_{\mathcal{D}} y\, dA, \qquad M_y = \iint_{\mathcal{D}} x\, dA$$

Show that $M_x = \oint_C xy\, dy$. Find a similar expression for M_y.

22. Use the result of Exercise 21 to compute the moments of the semicircle $x^2 + y^2 = R^2$, $y \ge 0$ as line integrals. Verify that the centroid is $(0, 4R/(3\pi))$.

23. Let C_R be the circle of radius R centered at the origin. Use the general form of Green's Theorem to determine $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is a vector field such that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 9$ and $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = x^2 + y^2$ for (x, y) in the annulus $1 \le x^2 + y^2 \le 4$.

24. Referring to Figure 24, suppose that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 12$. Use Green's Theorem to determine $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$, assuming that $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = -3$ in $\mathcal{D}$.

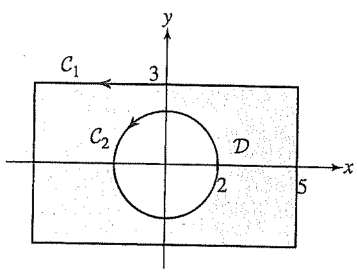


FIGURE 24

25. Referring to Figure 25, suppose that

$$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi, \qquad \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi$$

Use Green's Theorem to determine the circulation of $\mathbf{F}$ around C_1, assuming that $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 9$ on the shaded region.

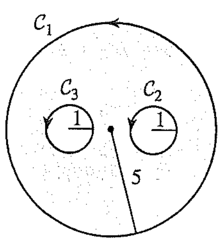


FIGURE 25

26. Let $\mathbf{F}$ be the vortex vector field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

In Section 16.3, we verified that $\int_{C_R} \mathbf{F} \cdot d\mathbf{r} = 2\pi$, where C_R is the circle of radius R centered at the origin. Prove that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any simple closed curve C whose interior contains the origin (Figure 26). *Hint:* Apply the general form of Green's Theorem to the domain between C and C_R, where R is so small that C_R is contained in C.

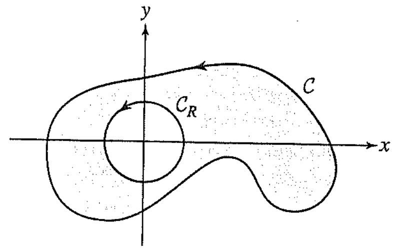


FIGURE 26

In Exercises 27–30, we refer to the integrand that occurs in Green's Theorem and that appears as

$$\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

27. For the vector fields (A)–(D) in Figure 27, state whether the curl_z at the origin appears to be positive, negative, or zero.

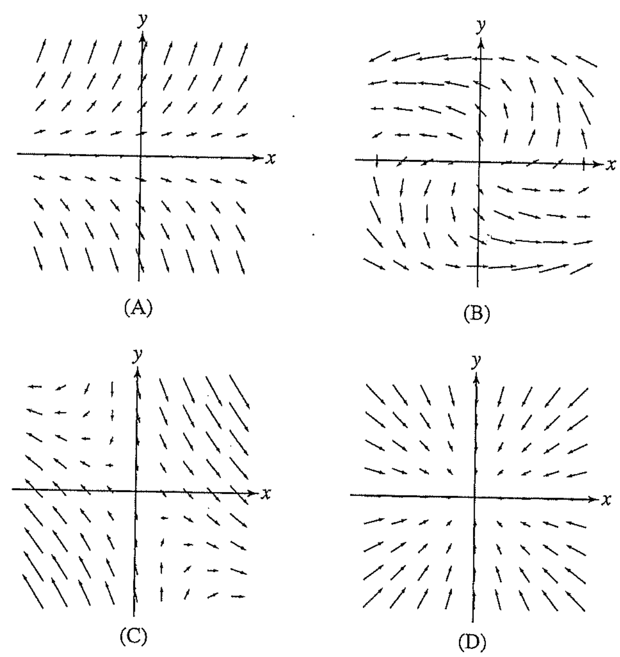


FIGURE 27

28. Estimate the circulation of a vector field $\mathbf{F}$ around a circle of radius $R = 0.1$, assuming that $\text{curl}_z(\mathbf{F})$ takes the value 4 at the center of the circle.

29. Estimate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x + 0.1y^2, y - 0.1x^2 \rangle$ and C encloses a small region of area 0.25 containing the point $P = (1, 1)$.

30. Let $\mathbf{F}$ be the velocity field. Estimate the circulation of $\mathbf{F}$ around a circle of radius $R = 0.05$ with center P, assuming that $\text{curl}_z(\mathbf{F})(P) = -3$. In which direction would a small paddle placed at P rotate? How fast would it rotate (in radians per second) if $\mathbf{F}$ is expressed in meters per second?

17.2 EXERCISES

Preliminary Questions

1. Indicate with an arrow the boundary orientation of the boundary curves of the surfaces in Figure 14, oriented by the outward-pointing normal vectors.

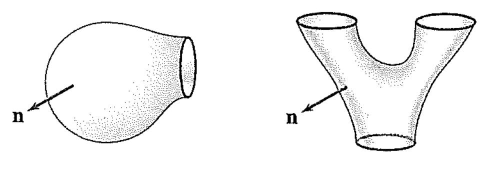


FIGURE 14

2. Let $\mathbf{F} = \text{curl}(\mathbf{A})$. Which of the following are related by Stokes' Theorem?

(a) The circulation of $\mathbf{A}$ and flux of $\mathbf{F}$

(b) The circulation of $\mathbf{F}$ and flux of $\mathbf{A}$

3. What is the definition of a vector potential?

4. Which of the following statements is correct?

(a) The flux of $\text{curl}(\mathbf{A})$ through every oriented surface is zero.

(b) The flux of $\text{curl}(\mathbf{A})$ through every closed, oriented surface is zero.

5. Which condition on $\mathbf{F}$ guarantees that the flux through $\mathcal{S}_1$ is equal to the flux through $\mathcal{S}_2$ for any two oriented surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with the same oriented boundary?

Exercises

In Exercises 1–4, verify Stokes' Theorem for the given vector field and surface, oriented with an upward-pointing normal.

1. $\mathbf{F} = \langle 2xy, x, y+z \rangle$, the surface $z = 1 - x^2 - y^2$ for $x^2 + y^2 \le 1$

2. $\mathbf{F} = \langle yz, 0, x \rangle$, the portion of the plane $\frac{x}{2} + \frac{y}{3} + z = 1$ where $x, y, z \ge 0$

3. $\mathbf{F} = \langle e^{y-z}, 0, 0 \rangle$, the square with vertices $(1,0,1)$, $(1,1,1)$, $(0,1,1)$, and $(0,0,1)$

4. $\mathbf{F} = \langle y, x, x^2 + y^2 \rangle$, the upper hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$

In Exercises 5–10, calculate curl(**F**) *and then apply Stokes' Theorem to compute the flux of* curl(**F**) *through the given surface using a line integral.*

5. $\mathbf{F} = \left\langle e^{z^2} - y, e^{z^3} + x, \cos(xz) \right\rangle$, the upper hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$ with outward-pointing normal

6. $\mathbf{F} = \left\langle x + y, z^2 - 4, x\sqrt{y^2+1} \right\rangle$, surface of the wedge-shaped box in Figure 15 (bottom included, top excluded) with outward-pointing normal

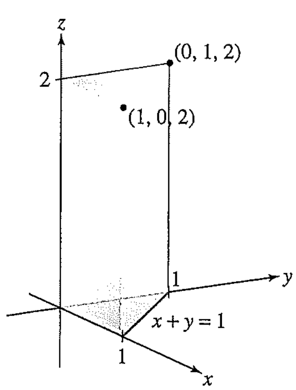


FIGURE 15

7. $\mathbf{F} = \langle 3z, 5x, -2y \rangle$, that part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$ with upward-pointing unit normal vector

8. $\mathbf{F} = \langle yz, -xz, z^3 \rangle$, that part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the two planes $z = 1$ and $z = 3$ with upward-pointing unit normal vector

9. $\mathbf{F} = \langle yz, xz, xy \rangle$, that part of the cylinder $x^2 + y^2 = 1$ that lies between the two planes $z = 1$ and $z = 4$ with outward-pointing unit normal vector

10. $\mathbf{F} = \langle 2y, e^z, -\arctan x \rangle$, that part of the paraboloid $z = 4 - x^2 - y^2$ cut off by the xy-plane with upward-pointing unit normal vector

In Exercises 11–14, apply Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *by finding the flux of* curl(**F**) *across an appropriate surface.*

11. $\mathbf{F} = \langle 3y, -2x, 3y \rangle$, C is the circle $x^2 + y^2 = 9$, $z = 2$, oriented counterclockwise as viewed from above.

12. $\mathbf{F} = \langle yz, xy, xz \rangle$, C is the square with vertices $(0,0,2)$, $(1,0,2)$, $(1,1,2)$, and $(0,1,2)$, oriented counterclockwise as viewed from above.

13. $\mathbf{F} = \langle y, z, x \rangle$, C is the triangle with vertices $(0,0,0)$, $(3,0,0)$, and $(0,3,3)$, oriented counterclockwise as viewed from above.

14. $\mathbf{F} = \langle y, -2z, 4x \rangle$, C is the boundary of that portion of the plane $x + 2y + 3z = 1$ that is in the first octant of space, oriented counterclockwise as viewed from above.

15. Let $\mathcal{S}$ be the surface of the cylinder (not including the top and bottom) of radius 2 for $1 \le z \le 6$, oriented with outward-pointing normal (Figure 16).

(a) Indicate with an arrow the orientation of $\partial\mathcal{S}$ (the top and bottom circles).

(b) Verify Stokes' Theorem for $\mathcal{S}$ and $\mathbf{F} = \langle yz^2, 0, 0 \rangle$.

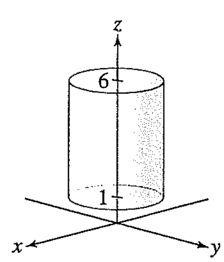


FIGURE 16

16. Let $\mathcal{S}$ be the portion of the plane $z = x$ contained in the half-cylinder of radius R depicted in Figure 17. Use Stokes' Theorem to calculate the circulation of $\mathbf{F} = \langle z, x, y + 2z \rangle$ around the boundary of $\mathcal{S}$ (a half-ellipse) in the counterclockwise direction when viewed from above. *Hint:* Show that curl($\mathbf{F}$) is orthogonal to the normal vector to the plane.

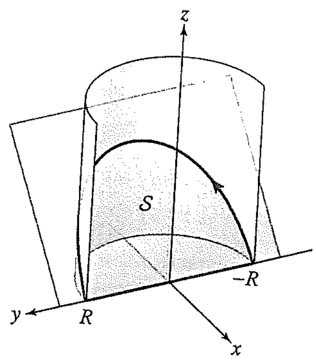


FIGURE 17

17. Let I be the flux of $\mathbf{F} = \left\langle e^y, 2xe^{x^2}, z^2 \right\rangle$ through the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \left\langle e^y, 2xe^{x^2}, 0 \right\rangle$. Find a vector field $\mathbf{A}$ such that curl($\mathbf{A}$) = $\mathbf{G}$.

(b) Use Stokes' Theorem to show that the flux of $\mathbf{G}$ through $\mathcal{S}$ is zero. *Hint:* Calculate the circulation of $\mathbf{A}$ around $\partial\mathcal{S}$.

(c) Calculate I. *Hint:* Use (b) to show that I is equal to the flux of $\langle 0, 0, z^2 \rangle$ through $\mathcal{S}$.

18. Let $\mathbf{F} = \langle 0, -z, 1 \rangle$. Let $\mathcal{S}$ be the spherical cap $x^2 + y^2 + z^2 \le 1$, where $z \ge \frac{1}{2}$. Evaluate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ directly as a surface integral. Then verify that $\mathbf{F} = \text{curl}(\mathbf{A})$, where $\mathbf{A} = \langle 0, x, xz \rangle$ and evaluate the surface integral again using Stokes' Theorem.

19. Let $\mathbf{A}$ be the vector potential and $\mathbf{B}$ the magnetic field of the infinite solenoid of radius R in Example 4. Use Stokes' Theorem to compute:

(a) The flux of $\mathbf{B}$ through a circle in the xy-plane of radius $r < R$

(b) The circulation of $\mathbf{A}$ around the boundary C of a surface lying outside the solenoid

20. The magnetic field $\mathbf{B}$ due to a small current loop (which we place at the origin) is called a **magnetic dipole** (Figure 18). Let $\rho = (x^2 + y^2 + z^2)^{1/2}$. For ρ large, $\mathbf{B} = \text{curl}(\mathbf{A})$, where

$$\mathbf{A} = \left\langle -\frac{y}{\rho^3}, \frac{x}{\rho^3}, 0 \right\rangle$$

(a) Let C be a horizontal circle of radius R with center $(0, 0, c)$, where c is large. Show that $\mathbf{A}$ is tangent to C.

(b) Use Stokes' Theorem to calculate the flux of $\mathbf{B}$ through C.

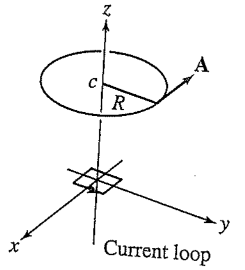


FIGURE 18

21. A uniform magnetic field $\mathbf{B}$ has constant strength b in the z-direction [i.e., $\mathbf{B} = \langle 0, 0, b \rangle$].

(a) Verify that $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ is a vector potential for $\mathbf{B}$, where $\mathbf{r} = \langle x, y, 0 \rangle$.

(b) Calculate the flux of $\mathbf{B}$ through the rectangle with vertices A, B, C, and D in Figure 19.

22. Let $\mathbf{F} = \langle -x^2 y, x, 0 \rangle$. Referring to Figure 19, let C be the closed path $ABCD$. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways. First, regard C as the boundary of the rectangle with vertices A, B, C, and D. Then treat C as the boundary of the wedge-shaped box with an open top.

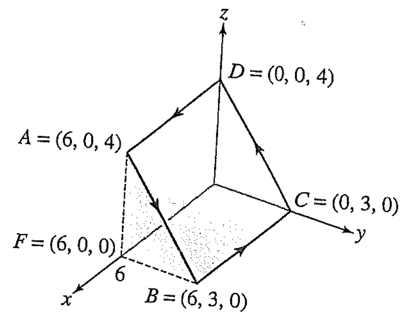


FIGURE 19

23. Let $\mathbf{F} = \langle y^2, 2z + x, 2y^2 \rangle$. Use Stokes' Theorem to find a plane with equation $ax + by + cz = 0$ (where a, b, c are not all zero) such that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed C lying in the plane. *Hint:* Choose a, b, c so that curl($\mathbf{F}$) lies in the plane.

24. Let $\mathbf{F} = \langle -z^2, 2zx, 4y - x^2 \rangle$, and let C be a simple closed curve in the plane $x + y + z = 4$ that encloses a region of area 16 (Figure 20). Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is oriented in the counterclockwise direction (when viewed from above the plane).

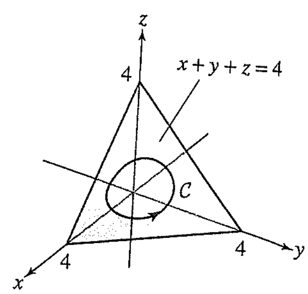


FIGURE 20

25. Let $\mathbf{F} = \langle y^2, x^2, z^2 \rangle$. Show that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two closed curves lying on a cylinder whose central axis is the z-axis (Figure 21).

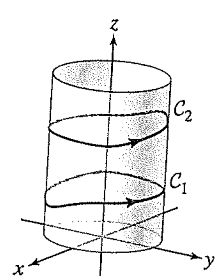


FIGURE 21

17.3 EXERCISES

Preliminary Questions

1. What is the flux of $\mathbf{F} = \langle 1, 0, 0 \rangle$ through a closed surface?

2. Justify the following statement: The flux of $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$ through every closed surface is positive.

3. Which of the following expressions are meaningful (where $\mathbf{F}$ is a vector field and f is a function)? Of those that are meaningful, which are automatically zero?

(a) $\operatorname{div}(\nabla f)$ (b) $\operatorname{curl}(\nabla f)$ (c) $\nabla\operatorname{curl}(f)$
(d) $\operatorname{div}(\operatorname{curl}(\mathbf{F}))$ (e) $\operatorname{curl}(\operatorname{div}(\mathbf{F}))$ (f) $\nabla(\operatorname{div}(\mathbf{F}))$

4. Which of the following statements is correct (where $\mathbf{F}$ is a continuously differentiable vector field defined everywhere)?

(a) The flux of $\operatorname{curl}(\mathbf{F})$ through all surfaces is zero.
(b) If $\mathbf{F} = \nabla\varphi$, then the flux of $\mathbf{F}$ through all surfaces is zero.
(c) The flux of $\operatorname{curl}(\mathbf{F})$ through all closed surfaces is zero.

5. How does the Divergence Theorem imply that the flux of $\mathbf{F} = \langle x^2, y - e^z, y - 2zx \rangle$ through a closed surface is equal to the enclosed volume?

Exercises

In Exercises 1–4, verify the Divergence Theorem for the vector field and region.

1. $\mathbf{F}(x, y, z) = \langle z, x, y \rangle$, the box $[0, 4] \times [0, 2] \times [0, 3]$

2. $\mathbf{F}(x, y, z) = \langle y, x, z \rangle$, the region $x^2 + y^2 + z^2 \le 4$

3. $\mathbf{F}(x, y, z) = \langle 2x, 3z, 3y \rangle$, the region $x^2 + y^2 \le 1, 0 \le z \le 2$

4. $\mathbf{F}(x, y, z) = \langle x, 0, 0 \rangle$, the region $x^2 + y^2 \le z \le 4$

In Exercises 5–16, use the Divergence Theorem to evaluate the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

5. $\mathbf{F}(x, y, z) = \langle 0, 0, z^3/3 \rangle$, $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = 1$.

6. $\mathbf{F}(x, y, z) = \langle y, z, x \rangle$, $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = 1$.

7. $\mathbf{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle$, $\mathcal{S}$ is the boundary of the cylinder given by $x^2 + y^2 \le 4, 0 \le z \le 3$.

8. $\mathbf{F}(x, y, z) = \langle x^2 z, yx, xyz \rangle$, $\mathcal{S}$ is the boundary of the tetrahedron given by $x + y + z \le 1, 0 \le x, 0 \le y, 0 \le z$.

9. $\mathbf{F}(x, y, z) = \langle x + z^2, xz + y^2, zx - y \rangle$, $\mathcal{S}$ is the surface that bounds the solid region with boundary given by the parabolic cylinder $z = 1 - x^2$, and the planes $z = 0$, $y = 0$ and $z + y = 5$.

10. $\mathbf{F}(x, y, z) = \langle zx, yx^3, x^2 z \rangle$, $\mathcal{S}$ is the surface that bounds the solid region with boundary given by $y = 4 - x^2 - z^2$ and $y = 0$.

11. $\mathbf{F}(x, y, z) = \langle x^3, 0, z^3 \rangle$, $\mathcal{S}$ is boundary of the region in the first octant of space given by $x^2 + y^2 + z^2 \le 4$, and $x \ge 0, y \ge 0, z \ge 0$.

12. $\mathbf{F}(x, y, z) = \langle e^{x+y}, e^{x+z}, e^{x+y} \rangle$, $\mathcal{S}$ is the boundary of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

13. $\mathbf{F}(x, y, z) = \langle x, y^2, z + y \rangle$, $\mathcal{S}$ is the boundary of the region contained in the cylinder $x^2 + y^2 = 4$ between the planes $z = x$ and $z = 8$.

14. $\mathbf{F}(x, y, z) = \langle x^2 - z^2, e^{z^2} - \cos x, y^3 \rangle$, $\mathcal{S}$ is the boundary of the region bounded by $x + 2y + 4z = 12$ and the coordinate planes in the first octant.

15. $\mathbf{F}(x, y, z) = \langle x + y, z, z - x \rangle$, $\mathcal{S}$ is the boundary of the region between the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

16. $\mathbf{F}(x, y, z) = \langle e^{z^2}, 2y + \sin(x^2 z), 4z + \sqrt{x^2 + 9y^2} \rangle$, $\mathcal{S}$ is the region $x^2 + y^2 \le z \le 8 - x^2 - y^2$.

17. Calculate the flux of the vector field $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + \mathbf{k}$ through the surface $\mathcal{S}$ in Figure 18. *Hint:* Apply the Divergence Theorem to the closed surface consisting of $\mathcal{S}$ and the unit disk.

18. Let $\mathcal{S}_1$ be the closed surface consisting of $\mathcal{S}$ in Figure 18 together with the unit disk. Find the volume enclosed by $\mathcal{S}_1$, assuming that

$$\iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot d\mathbf{S} = 72 \quad .$$

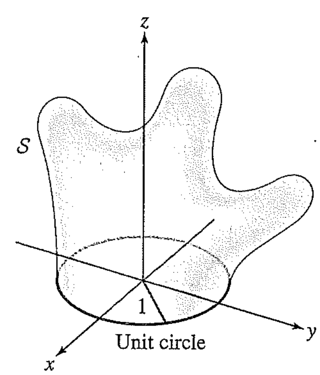


FIGURE 18 Surface $\mathcal{S}$ whose boundary is the unit circle.

19. Let $\mathcal{S}$ be the half-cylinder $x^2 + y^2 = 1, x \ge 0, 0 \le z \le 1$. Assume that $\mathbf{F}$ is a horizontal vector field (the z-component is zero) such that $\mathbf{F}(0, y, z) = zy^2\mathbf{i}$. Let $\mathcal{W}$ be the solid region enclosed by $\mathcal{S}$, and assume that

$$\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = 4$$

Find the flux of $\mathbf{F}$ through the curved side of $\mathcal{S}$.

20. Volume as a Surface Integral Let $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$. Prove that if $\mathcal{W}$ is a region in $\mathbf{R}^3$ with a smooth boundary $\mathcal{S}$, then

$$\text{Volume}(\mathcal{W}) = \frac{1}{3} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \qquad \boxed{9}$$

21. Use Eq. (9) to calculate the volume of the unit ball as a surface integral over the unit sphere.

22. Verify that Eq. (9) applied to the box $[0, a] \times [0, b] \times [0, c]$ yields the volume $V = abc$.

23. Let $\mathcal{W}$ be the region in Figure 19 bounded by the cylinder $x^2 + y^2 = 4$, the plane $z = x + 1$, and the xy-plane. Use the Divergence Theorem to compute the flux of $\mathbf{F}(x, y, z) = \langle z, x, y + z^2 \rangle$ through the boundary of $\mathcal{W}$.

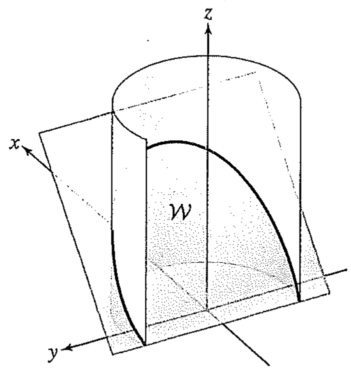


FIGURE 19

24. Let $I = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \left\langle \frac{2yz}{r^2}, -\frac{xz}{r^2}, -\frac{xy}{r^2} \right\rangle$$

($r = \sqrt{x^2 + y^2 + z^2}$) and $\mathcal{S}$ is the boundary of a region $\mathcal{W}$.

(a) Check that $\mathbf{F}$ is divergence-free.

(b) Show that $I = 0$ if $\mathcal{S}$ is a sphere centered at the origin. Explain, however, why the Divergence Theorem cannot be used to prove this.

25. The velocity field of a fluid $\mathbf{v}$ (in meters per second) has divergence $\text{div}(\mathbf{v})(P) = 3$ at the point $P = (2, 2, 2)$. Estimate the flow rate out of the sphere of radius 0.5 centered at P.

26. A hose feeds into a small screen box of volume 10 cm^3 that is suspended in a swimming pool. Water flows across the surface of the box at a rate of 12 cm^3/s. Estimate $\text{div}(\mathbf{v})(P)$, where $\mathbf{v}$ is the velocity field of the water in the pool and P is the center of the box. What are the units of $\text{div}(\mathbf{v})(P)$?

27. The electric field due to a unit electric dipole oriented in the $\mathbf{k}$-direction is $\mathbf{E} = \nabla(z/r^3)$, where $r = (x^2 + y^2 + z^2)^{1/2}$ (Figure 20). Let $\mathbf{e}_r = r^{-1}\langle x, y, z \rangle$.

(a) Show that $\mathbf{E} = r^{-3}\mathbf{k} - 3zr^{-4}\mathbf{e}_r$.

(b) Calculate the flux of $\mathbf{E}$ through a sphere centered at the origin.

(c) Calculate $\text{div}(\mathbf{E})$.

(d) Can we use the Divergence Theorem to compute the flux of $\mathbf{E}$ through a sphere centered at the origin?

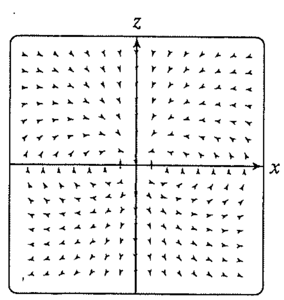


FIGURE 20 The dipole vector field restricted to the xz-plane.

28. Let $\mathbf{E}$ be the electric field due to a long, uniformly charged rod of radius R with charge density δ per unit length (Figure 21). By symmetry, we may assume that $\mathbf{E}$ is everywhere perpendicular to the rod and its magnitude $E(d)$ depends only on the distance d to the rod (strictly speaking, this would hold only if the rod were infinite, but it is nearly true if the rod is long enough). Show that $E(d) = \delta/2\pi\epsilon_0 d$ for $d > R$. *Hint:* Apply Gauss's Law to a cylinder of radius R and of unit length with its axis along the rod.

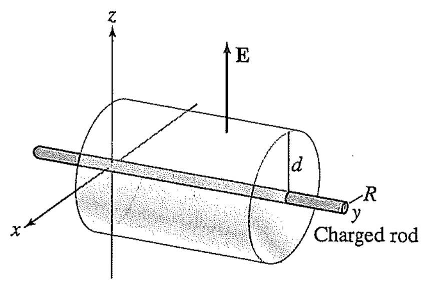


FIGURE 21

29. Let $\mathcal{W}$ be the region between the sphere of radius 4 and the cube of side 1, both centered at the origin. What is the flux through the boundary $\mathcal{S} = \partial\mathcal{W}$ of a vector field $\mathbf{F}$ whose divergence has the constant value $\text{div}(\mathbf{F}) = -4$?

30. Let $\mathcal{W}$ be the region between the sphere of radius 3 and the sphere of radius 2, both centered at the origin. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = x\mathbf{i}$ through the boundary $\mathcal{S} = \partial\mathcal{W}$.

31. Find and prove a Product Rule expressing $\text{div}(f\mathbf{F})$ in terms of $\text{div}(\mathbf{F})$ and ∇f.

32. Prove the identity

$$\text{div}(\mathbf{F} \times \mathbf{G}) = \text{curl}(\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot \text{curl}(\mathbf{G})$$

Then prove that the cross product of two irrotational vector fields is incompressible [$\mathbf{F}$ is called **irrotational** if $\text{curl}(\mathbf{F}) = 0$ and **incompressible** if $\text{div}(\mathbf{F}) = 0$].

33. Prove that $\text{div}(\nabla f \times \nabla g) = 0$.

In Exercises 34–36, Δ denotes the Laplace operator defined by

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

34. Prove the identity

$$\text{curl}(\text{curl}(\mathbf{F})) = \nabla(\text{div}(\mathbf{F})) - \Delta\mathbf{F}$$

where $\Delta\mathbf{F}$ denotes $\langle \Delta F_1, \Delta F_2, \Delta F_3 \rangle$.

35. A function φ satisfying $\Delta\varphi = 0$ is called **harmonic**.

(a) Show that $\Delta\varphi = \text{div}(\nabla\varphi)$ for any function φ.

(b) Show that φ is harmonic if and only if $\text{div}(\nabla\varphi) = 0$.

(c) Show that if $\mathbf{F}$ is the gradient of a harmonic function, then $\text{curl}(F) = 0$ and $\text{div}(F) = 0$.

(d) Show that $\mathbf{F}(x, y, z) = \langle xz, -yz, \frac{1}{2}(x^2 - y^2) \rangle$ is the gradient of a harmonic function. What is the flux of $\mathbf{F}$ through a closed surface?

36. Let $\mathbf{F} = r^n\mathbf{e}_r$, where n is any number, $r = (x^2 + y^2 + z^2)^{1/2}$, and $\mathbf{e}_r = r^{-1}\langle x, y, z \rangle$ is the unit radial vector.

(a) Calculate $\text{div}(\mathbf{F})$.

(b) Calculate the flux of $\mathbf{F}$ through the surface of a sphere of radius R centered at the origin. For which values of n is this flux independent of R?