6. For which of the following functions is the double integral over the rectangle in Figure 15 equal to zero? Explain your reasoning.

(a) $f(x, y) = x^2y$ (b) $f(x, y) = xy^2$

(c) $f(x, y) = \sin x$ (d) $f(x, y) = e^x$

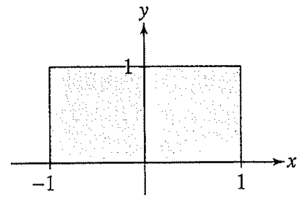


FIGURE 15

Exercises

1. Compute the Riemann sum $S_{4,3}$ to estimate the double integral of $f(x, y) = xy$ over $\mathcal{R} = [1, 3] \times [1, 2.5]$. Use the regular partition and upper-right vertices of the subrectangles as sample points.

2. Compute the Riemann sum with $N = M = 2$ to estimate the integral of $\sqrt{x+y}$ over $\mathcal{R} = [0, 1] \times [0, 1]$. Use the regular partition and midpoints of the subrectangles as sample points.

In Exercises 3–6, compute the Riemann sums for the double integral $\iint_{\mathcal{R}} f(x, y)\, dA$, *where* $\mathcal{R} = [1, 4] \times [1, 3]$, *for the grid and two choices of sample points shown in Figure 16.*

3. $f(x, y) = 2x + y$

4. $f(x, y) = 7$

5. $f(x, y) = 4x$

6. $f(x, y) = x - 2y$

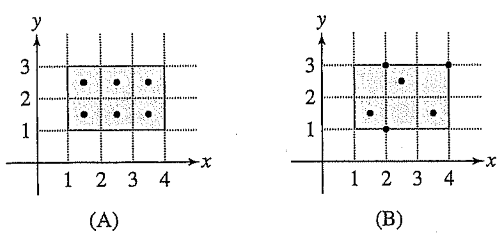


FIGURE 16

7. Let $\mathcal{R} = [0, 1] \times [0, 1]$. Estimate $\iint_{\mathcal{R}} (x + y)\, dA$ by computing two different Riemann sums, each with at least six rectangles.

8. Evaluate $\iint_{\mathcal{R}} 4\, dA$, where $\mathcal{R} = [2, 5] \times [4, 7]$.

9. Evaluate $\iint_{\mathcal{R}} (15 - 3x)\, dA$, where $\mathcal{R} = [0, 5] \times [0, 3]$, and sketch the corresponding solid region (see Example 2).

10. Evaluate $\iint_{\mathcal{R}} (-5)\, dA$, where $\mathcal{R} = [2, 5] \times [4, 7]$.

11. The following table gives the approximate height at quarter-meter intervals of a mound of gravel. Estimate the volume of the mound by computing the average of the two Riemann sums $S_{4,3}$ with lower-left and upper-right vertices of the subrectangles as sample points.

0.75	0.1	0.2	0.2	0.15	0.1
0.5	0.2	0.3	0.5	0.4	0.2
0.25	0.15	0.2	0.4	0.3	0.2
0	0.1	0.15	0.2	0.15	0.1
y / x	0	0.25	0.5	0.75	1

12. Use the following table to compute a Riemann sum $S_{3,3}$ for $f(x, y)$ on the square $\mathcal{R} = [0, 1.5] \times [0.5, 2]$. Use the regular partition and sample points of your choosing.

Values of $f(x, y)$

2	2.6	2.17	1.86	1.62	1.44
1.5	2.2	1.83	1.57	1.37	1.22
1	1.8	1.5	1.29	1.12	1
0.5	1.4	1.17	1	0.87	0.78
0	1	0.83	0.71	0.62	0.56
y / x	0	0.5	1	1.5	2

13. CAS Let $S_{N,N}$ be the Riemann sum for $\int_0^1 \int_0^1 e^{x^3-y^3}\, dy\, dx$ using the regular partition and the lower left-hand vertex of each subrectangle as sample points. Use a computer algebra system to calculate $S_{N,N}$ for $N = 25, 50, 100$.

14. CAS Let $S_{N,M}$ be the Riemann sum for

$$\int_0^4 \int_0^2 \ln(1 + x^2 + y^2)\, dy\, dx$$

using the regular partition and the upper right-hand vertex of each subrectangle as sample points. Use a computer algebra system to calculate $S_{2N,N}$ for $N = 25, 50, 100$.

In Exercises 15–18, use symmetry to evaluate the double integral.

15. $\iint_{\mathcal{R}} x^3\, dA, \quad \mathcal{R} = [-4, 4] \times [0, 5]$

16. $\iint_{\mathcal{R}} 1\, dA, \quad \mathcal{R} = [2, 4] \times [-7, 7]$

17. $\iint_{\mathcal{R}} \sin x\, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi]$

18. $\iint_{\mathcal{R}} (2 + x^2y)\, dA, \quad \mathcal{R} = [0, 1] \times [-1, 1]$

In Exercises 19–36, evaluate the iterated integral.

19. $\int_1^3 \int_0^2 x^3y\, dy\, dx$

20. $\int_0^2 \int_1^3 x^3y\, dx\, dy$

21. $\int_4^9 \int_{-3}^8 1\, dx\, dy$

22. $\int_{-4}^{-1} \int_4^8 (-5)\, dx\, dy$

23. $\int_{-1}^1 \int_0^{\pi} x^2 \sin y\, dy\, dx$

24. $\int_{-1}^1 \int_0^{\pi} x^2 \sin y\, dx\, dy$

25. $\int_2^6 \int_1^4 x^2\, dx\, dy$

26. $\int_2^6 \int_1^4 y^2\, dx\, dy$

27. $\int_0^1 \int_0^2 (x + 4y^3)\, dx\, dy$

28. $\int_0^2 \int_0^2 (x^2 - y^2)\, dy\, dx$

29. $\int_0^4 \int_0^9 \sqrt{x + 4y}\, dx\, dy$

30. $\displaystyle\int_0^{\pi/4}\int_{\pi/4}^{\pi/2} \cos(2x+y)\,dy\,dx$

31. $\displaystyle\int_1^2\int_0^4 \frac{dy\,dx}{x+y}$

32. $\displaystyle\int_1^2\int_2^4 e^{3x-y}\,dy\,dx$

33. $\displaystyle\int_0^4\int_0^5 \frac{dy\,dx}{\sqrt{x+y}}$

34. $\displaystyle\int_0^8\int_1^2 \frac{x\,dx\,dy}{\sqrt{x^2+y}}$

35. $\displaystyle\int_1^2\int_1^3 \frac{\ln(xy)\,dy\,dx}{y}$

36. $\displaystyle\int_0^1\int_2^3 \frac{1}{(x+4y)^3}\,dx\,dy$

In Exercises 37–42, evaluate the integral.

37. $\displaystyle\iint_{\mathcal{R}} \frac{x}{y}\,dA, \quad \mathcal{R} = [-2,4]\times[1,3]$

38. $\displaystyle\iint_{\mathcal{R}} x^2y\,dA, \quad \mathcal{R} = [-1,1]\times[0,2]$

39. $\displaystyle\iint_{\mathcal{R}} \cos x \sin 2y\,dA, \quad \mathcal{R} = \left[0,\tfrac{\pi}{2}\right]\times\left[0,\tfrac{\pi}{2}\right]$

40. $\displaystyle\iint_{\mathcal{R}} \frac{y}{x+1}\,dA, \quad \mathcal{R} = [0,2]\times[0,4]$

41. $\displaystyle\iint_{\mathcal{R}} e^x \sin y\,dA, \quad \mathcal{R} = [0,2]\times\left[0,\tfrac{\pi}{4}\right]$

42. $\displaystyle\iint_{\mathcal{R}} e^{3x+4y}\,dA, \quad \mathcal{R} = [0,1]\times[1,2]$

43. Let $f(x,y) = mxy^2$, where m is a constant. Find a value of m such that $\displaystyle\iint_{\mathcal{R}} f(x,y)\,dA = 1$, where $\mathcal{R} = [0,1]\times[0,2]$.

44. Evaluate $I = \displaystyle\int_1^3\int_0^1 ye^{xy}\,dy\,dx$. You will need Integration by Parts and the formula

$$\int e^x(x^{-1} - x^{-2})\,dx = x^{-1}e^x + C$$

Then evaluate I again using Fubini's Theorem to change the order of integration (i.e., integrate first with respect to x). Which method is easier?

45. Evaluate $\displaystyle\int_0^1\int_0^1 y\sqrt{1+xy}\,dy\,dx$. *Hint:* Change the order of integration.

46. Evaluate $\displaystyle\int_0^1\int_0^1 xe^{xy}\,dx\,dy$. *Hint:* Change the order of integration.

47. Evaluate $\displaystyle\int_0^1\int_0^1 \frac{y}{1+xy}\,dy\,dx$. *Hint:* Change the order of integration.

48. Calculate a Riemann sum $S_{3,3}$ on the square $\mathcal{R} = [0,3]\times[0,3]$ for the function $f(x,y)$ whose contour plot is shown in Figure 17. Choose sample points and use the plot to find the values of $f(x,y)$ at these points.

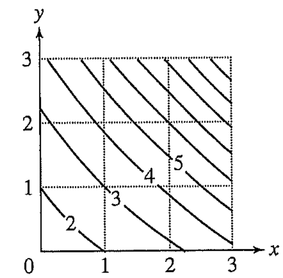


FIGURE 17 Contour plot of $f(x,y)$.

49. Using Fubini's Theorem, argue that the solid in Figure 18 has volume AL, where A is the area of the front face of the solid.

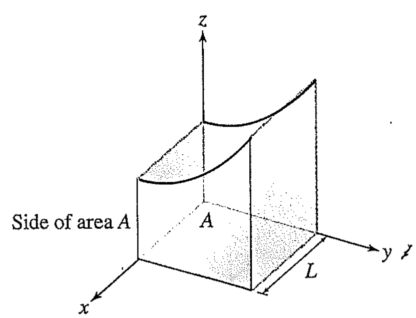


FIGURE 18

Further Insights and Challenges

50. Prove the following extension of the Fundamental Theorem of Calculus to two variables: If $\dfrac{\partial^2 F}{\partial x\,\partial y} = f(x,y)$, then

$$\iint_{\mathcal{R}} f(x,y)\,dA = F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

where $\mathcal{R} = [a,b]\times[c,d]$.

51. Let $F(x,y) = x^{-1}e^{xy}$. Show that $\dfrac{\partial^2 F}{\partial x\,\partial y} = ye^{xy}$ and use the result of Exercise 50 to evaluate $\displaystyle\iint_{\mathcal{R}} ye^{xy}\,dA$ for the rectangle $\mathcal{R} = [1,3]\times[0,1]$.

52. Find a function $F(x,y)$ satisfying $\dfrac{\partial^2 F}{\partial x\,\partial y} = 6x^2y$ and use the result of Exercise 50 to evaluate $\displaystyle\iint_{\mathcal{R}} 6x^2y\,dA$ for the rectangle $\mathcal{R} = [0,1]\times[0,4]$.

53. In this exercise, we use double integration to evaluate the following improper integral for $a > 0$ a positive constant:

$$I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\,dx$$

(a) Use L'Hôpital's Rule to show that $f(x) = \dfrac{e^{-x} - e^{-ax}}{x}$, though not defined at $x = 0$, can be made continuous by assigning the value $f(0) = a - 1$.

(b) Prove that $|f(x)| \le e^{-x} + e^{-ax}$ for $x > 1$ (use the triangle inequality), and apply the Comparison Theorem to show that $I(a)$ converges.

(c) Show that $I(a) = \displaystyle\int_0^\infty\int_1^a e^{-xy}\,dy\,dx$.

Exercises

1. Calculate the Riemann sum for $f(x, y) = x - y$ and the shaded domain $\mathcal{D}$ in Figure 21 with two choices of sample points, • and ∘. Which do you think is a better approximation to the integral of f over $\mathcal{D}$? Why?

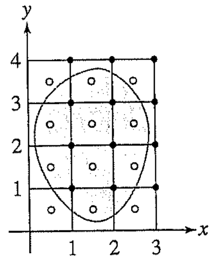


FIGURE 21

2. Approximate values of $f(x, y)$ at sample points on a grid are given in Figure 22. Estimate $\iint_{\mathcal{D}} f(x, y)\, dx\, dy$ for the shaded domain by computing the Riemann sum with the given sample points.

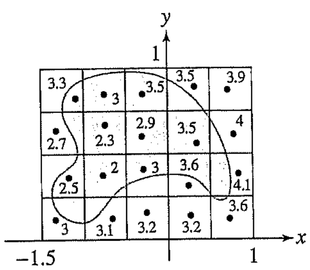


FIGURE 22

3. Express the domain $\mathcal{D}$ in Figure 23 as both a vertically simple region and a horizontally simple region, and evaluate the integral of $f(x, y) = xy$ over $\mathcal{D}$ as an iterated integral in two ways.

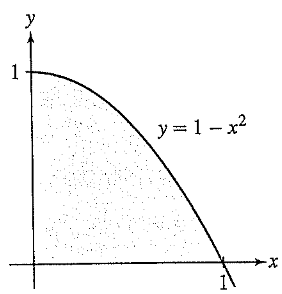


FIGURE 23

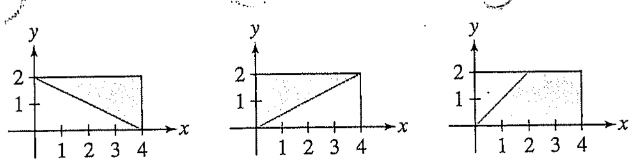
4. Sketch the domain

$$\mathcal{D}: 0 \le x \le 1, \quad x^2 \le y \le 4 - x^2$$

and evaluate $\iint_{\mathcal{D}} y\, dA$ as an iterated integral.

In Exercises 5–7, compute the double integral of $f(x, y) = x^2 y$ over the given shaded domain in Figure 24.

5. (A) **6.** (B) **7.** (C)



(A) (B) (C)

FIGURE 24

8. Sketch the domain $\mathcal{D}$ defined by $x + y \le 12$, $x \ge 4$, $y \ge 4$ and compute $\iint_{\mathcal{D}} e^{x+y}\, dA$.

9. Integrate $f(x, y) = x$ over the region bounded by $y = x^2$ and $y = x + 2$.

10. Sketch the region $\mathcal{D}$ between $y = x^2$ and $y = x(1 - x)$. Express $\mathcal{D}$ as a simple region and calculate the integral of $f(x, y) = 2y$ over $\mathcal{D}$.

11. Evaluate $\iint_{\mathcal{D}} \frac{y}{x}\, dA$, where $\mathcal{D}$ is the shaded part of the semicircle of radius 2 in Figure 25.

12. Calculate the double integral of $f(x, y) = y^2$ over the rhombus $\mathcal{R}$ in Figure 26.

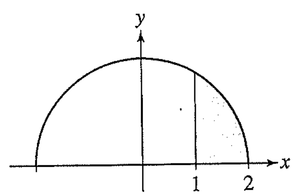


FIGURE 25 $y = \sqrt{4 - x^2}$.

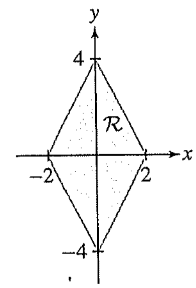


FIGURE 26
$|x| + \frac{1}{2}|y| \le 1$.

13. Calculate the double integral of $f(x, y) = x + y$ over the domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \le 4,\ y \ge 0\}$.

14. Integrate $f(x, y) = (x + y + 1)^{-2}$ over the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 8)$.

15. Calculate the integral of $f(x, y) = x$ over the region $\mathcal{D}$ bounded above by $y = x(2 - x)$ and below by $x = y(2 - y)$. *Hint:* Apply the quadratic formula to the lower boundary curve to solve for y as a function of x.

16. Integrate $f(x, y) = x$ over the region bounded by $y = x$, $y = 4x - x^2$, and $y = 0$ in two ways: as a vertically simple region and as a horizontally simple region.

In Exercises 17–24, compute the double integral of $f(x, y)$ over the domain $\mathcal{D}$ indicated.

17. $f(x, y) = x^2 y$; $\quad 1 \le x \le 3, \quad x \le y \le 2x + 1$

18. $f(x, y) = 1$; $\quad 0 \le x \le 1, \quad 1 \le y \le e^x$

19. $f(x, y) = x$; $\quad 0 \le x \le 1, \quad 1 \le y \le e^{x^2}$

20. $f(x, y) = \cos(2x + y)$; $\quad \frac{1}{2} \le x \le \frac{\pi}{2}, \quad 1 \le y \le 2x$

21. $f(x, y) = 2xy$; bounded by $x = y$, $x = y^2$

22. $f(x, y) = \sin x$; bounded by $x = 0$, $x = 1$, $y = \cos x$

23. $f(x, y) = e^{x+y}$; bounded by $y = x - 1$, $y = 12 - x$ for $2 \le y \le 4$

24. $f(x, y) = (x + y)^{-1}$; bounded by $y = x$, $y = 1$, $y = e$, $x = 0$

In Exercises 25–28, sketch the domain of integration and express as an iterated integral in the opposite order.

25. $\int_0^4 \int_x^4 f(x, y)\, dy\, dx$

26. $\int_4^9 \int_{\sqrt{y}}^3 f(x, y)\, dx\, dy$

27. $\int_4^9 \int_2^{\sqrt{y}} f(x, y)\, dx\, dy$

28. $\int_0^1 \int_{e^x}^e f(x, y)\, dy\, dx$

29. Sketch the domain $\mathcal{D}$ corresponding to

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2 + 5y}\, dx\, dy$$

Then change the order of integration and evaluate.

30. Change the order of integration and evaluate

$$\int_0^1 \int_0^{\pi/2} x \cos(xy)\, dx\, dy$$

Explain the simplification achieved by changing the order.

31. Compute the integral of $f(x, y) = (\ln y)^{-1}$ over the domain $\mathcal{D}$ bounded by $y = e^x$ and $y = e^{\sqrt{x}}$. *Hint:* Choose the order of integration that enables you to evaluate the integral.

32. Evaluate by changing the order of integration:
$$\int_0^4 \int_{\sqrt{x}}^2 \sin y^3\, dy\, dx$$

In Exercises 33–36, sketch the domain of integration. Then change the order of integration and evaluate. Explain the simplification achieved by changing the order.

33. $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$

34. $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3 + 1}\, dx\, dy$

35. $\int_0^1 \int_{y=x}^1 xe^{y^3}\, dy\, dx$

36. $\int_0^1 \int_{y=x^{2/3}}^1 xe^{y^4}\, dy\, dx$

37. Sketch the domain $\mathcal{D}$ where $0 \le x \le 2$, $0 \le y \le 2$, and x *or* y is greater than 1. Then compute $\iint_{\mathcal{D}} e^{x+y}\, dA$.

38. Calculate $\iint_{\mathcal{D}} e^x\, dA$, where $\mathcal{D}$ is bounded by the lines $y = x + 1$, $y = x$, $x = 0$, and $x = 1$.

In Exercises 39–42, calculate the double integral of $f(x, y)$ over the triangle indicated in Figure 27.

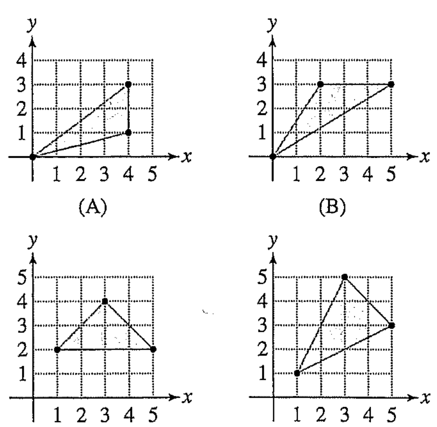


FIGURE 27

39. $f(x, y) = e^{x^2}$, (A)

40. $f(x, y) = 1 - 2x$, (B)

41. $f(x, y) = \frac{x}{y^2}$, (C)

42. $f(x, y) = x + 1$, (D)

43. Calculate the double integral of $f(x, y) = \frac{\sin y}{y}$ over the region $\mathcal{D}$ in Figure 28.

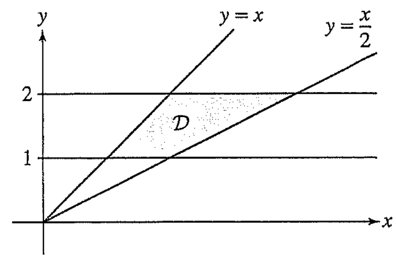


FIGURE 28

44. Evaluate $\iint_{\mathcal{D}} x\, dA$ for $\mathcal{D}$ in Figure 29.

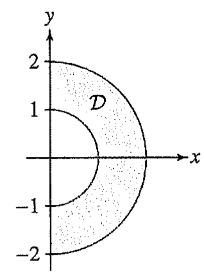


FIGURE 29

45. Find the volume of the region bounded by $z = 40 - 10y$, $z = 0$, $y = 0$, and $y = 4 - x^2$.

46. Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \le x \le 2$.

47. Find the volume of the region bounded by $z = 16 - y$, $z = y$, $y = x^2$, and $y = 8 - x^2$.

48. Find the volume of the region bounded by $y = 1 - x^2$, $z = 1$, $y = 0$, and $z + y = 2$.

49. Set up a double integral that gives the volume of the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. (Do not evaluate the double integral.)

50. Set up a double integral that gives the volume of the region bounded by $z = 1 - y^2$, $z = y$, $x = 0$, $y = 0$, and $x + y = 1$. (Do not evaluate the double integral.)

51. Calculate the average value of $f(x, y) = e^{x+y}$ on the square $[0, 1] \times [0, 1]$.

52. Calculate the average height above the x-axis of a point in the region $0 \le x \le 1$, $0 \le y \le x^2$.

53. Find the average height of the "ceiling" in Figure 30 defined by $z = y^2 \sin x$ for $0 \le x \le \pi$, $0 \le y \le 1$.

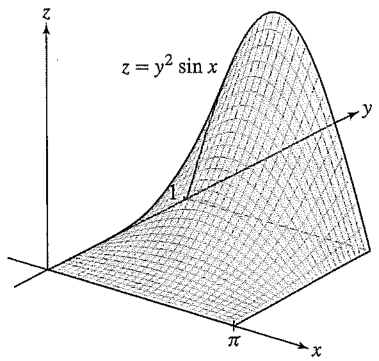


FIGURE 30

54. Calculate the average value of the x-coordinate of a point on the semicircle $x^2 + y^2 \le R^2$, $x \ge 0$. What is the average value of the y-coordinate?

55. What is the average value of the linear function

$$f(x, y) = mx + ny + p$$

on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1$? Argue by symmetry rather than calculation.

56. Find the average square distance from the origin to a point in the domain $\mathcal{D}$ in Figure 31.

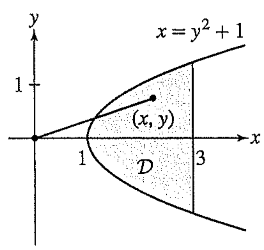


FIGURE 31

57. Let $\mathcal{D}$ be the rectangle $0 \le x \le 2$, $-\frac{1}{8} \le y \le \frac{1}{8}$, and let $f(x, y) = \sqrt{x^3 + 1}$. Prove that

$$\iint_{\mathcal{D}} f(x, y)\, dA \le \frac{3}{2}$$

58. (a) Use the inequality $0 \le \sin x \le x$ for $x \ge 0$ to show that

$$\int_0^1 \int_0^1 \sin(xy)\, dx\, dy \le \frac{1}{4}$$

(b) Use a computer algebra system to evaluate the double integral to three decimal places.

59. Prove the inequality $\displaystyle\iint_{\mathcal{D}} \frac{dA}{4 + x^2 + y^2} \le \pi$, where $\mathcal{D}$ is the disk $x^2 + y^2 \le 4$.

60. Let $\mathcal{D}$ be the domain bounded by $y = x^2 + 1$ and $y = 2$. Prove t inequality

$$\frac{4}{3} \le \iint_{\mathcal{D}} (x^2 + y^2)\, dA \le \frac{20}{3}$$

61. Let $\overline{f}$ be the average of $f(x, y) = xy^2$ on $\mathcal{D} = [0, 1] \times [0, 4]$. Find a point $P \in \mathcal{D}$ such that $f(P) = \overline{f}$ (the existence of such a point is guaranteed by the Mean Value Theorem for Double Integrals).

62. Verify the Mean Value Theorem for Double Integrals for $f(x, y) = e^{x-y}$ on the triangle bounded by $y = 0$, $x = 1$, and $y = x$.

In Exercises 63 and 64, use (11) to estimate the double integral.

63. The following table lists the areas of the subdomains $\mathcal{D}_j$ of the domain $\mathcal{D}$ in Figure 32 and the values of a function $f(x, y)$ at sample points $P_j \in \mathcal{D}_j$. Estimate $\displaystyle\iint_{\mathcal{D}} f(x, y)\, dA$.

j	1	2	3	4	5	6
Area($\mathcal{D}_j$)	1.2	1.1	1.4	0.6	1.2	0.8
$f(P_j)$	9	9.1	9.3	9.1	8.9	8.8

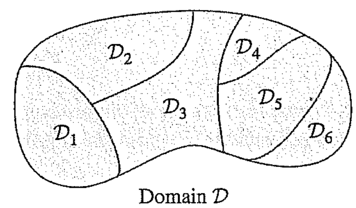


FIGURE 32

64. The domain $\mathcal{D}$ between the circles of radii 5 and 5.2 in the first quadrant in Figure 33 is divided into six subdomains of angular width $\Delta\theta = \frac{\pi}{12}$, and the values of a function $f(x, y)$ at sample points are given. Compute the area of the subdomains and estimate $\displaystyle\iint_{\mathcal{D}} f(x, y)\, dA$.

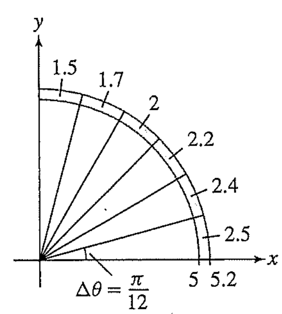


FIGURE 33

65. According to Eq. (3), the area of a domain $\mathcal{D}$ is equal to $\displaystyle\iint_{\mathcal{D}} 1\, dA$. Prove that if $\mathcal{D}$ is the region between two curves $y = g_1(x)$ and $y = g_2(x)$ with $g_2(x) \le g_1(x)$ for $a \le x \le b$, then

$$\iint_{\mathcal{D}} 1\, dA = \int_a^b (g_1(x) - g_2(x))\, dx$$

The iterated integral may be written in any one of six possible orders—for example,

$$\int_{z=p}^{q}\int_{y=c}^{d}\int_{x=a}^{b} f(x, y, z)\,dx\,dy\,dz$$

- A *z-simple region* $\mathcal{W}$ in $\mathbf{R}^3$ is a region consisting of the points (x, y, z) between two surfaces $z = z_1(x, y)$ and $z = z_2(x, y)$, where $z_1(x, y) \le z_2(x, y)$, lying over a domain $\mathcal{D}$ in the xy-plane. In other words, $\mathcal{W}$ is defined by

$$(x, y) \in \mathcal{D}, \qquad z_1(x, y) \le z \le z_2(x, y)$$

Similarly, we have x-simple regions and y-simple regions.
- The triple integral over $\mathcal{W}$ is equal to an iterated integral:

$$\iiint_{\mathcal{W}} f(x, y, z)\,dV = \iint_{\mathcal{D}} \left(\int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\right) dA$$

- The *average value* of $f(x, y, z)$ on a region $\mathcal{W}$ of volume V is the quantity

$$\overline{f} = \frac{1}{V}\iiint_{\mathcal{W}} f(x, y, z)\,dV, \qquad V = \iiint_{\mathcal{W}} 1\,dV$$

15.3 EXERCISES

Preliminary Questions

1. Which of (a)–(c) is not equal to $\int_0^1\int_3^4\int_6^7 f(x, y, z)\,dz\,dy\,dx$?

(a) $\int_6^7\int_0^1\int_3^4 f(x, y, z)\,dy\,dx\,dz$

(b) $\int_3^4\int_0^1\int_6^7 f(x, y, z)\,dz\,dx\,dy$

(c) $\int_0^1\int_3^4\int_6^7 f(x, y, z)\,dx\,dz\,dy$

2. Which of the following is not a meaningful triple integral?

(a) $\int_0^1\int_0^x\int_{x+y}^{2x+y} e^{x+y+z}\,dz\,dy\,dx$

(b) $\int_0^1\int_0^z\int_{x+y}^{2x+y} e^{x+y+z}\,dz\,dy\,dx$

3. Describe the projection of the region of integration $\mathcal{W}$ onto the xy-plane:

(a) $\int_0^1\int_0^x\int_0^{x^2+y^2} f(x, y, z)\,dz\,dy\,dx$

(b) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_2^4 f(x, y, z)\,dz\,dy\,dx$

Exercises

In Exercises 1–8, evaluate $\iiint_{\mathcal{B}} f(x, y, z)\,dV$ *for the specified function* f *and box* $\mathcal{B}$.

1. $f(x, y, z) = z^4;\quad 2 \le x \le 8,\quad 0 \le y \le 5,\quad 0 \le z \le 1$

2. $f(x, y, z) = xz^2;\quad [-2, 3] \times [1, 3] \times [1, 4]$

3. $f(x, y, z) = xe^{y-2z};\quad 0 \le x \le 2,\quad 0 \le y \le 1,\quad 0 \le z \le 1$

4. $f(x, y, z) = \dfrac{x}{(y+z)^2};\quad [0, 2] \times [2, 4] \times [-1, 1]$

5. $f(x, y, z) = (x - y)(y - z);\quad [0, 1] \times [0, 3] \times [0, 3]$

6. $f(x, y, z) = \dfrac{z}{x};\quad 1 \le x \le 3,\quad 0 \le y \le 2,\quad 0 \le z \le 4$

7. $f(x, y, z) = (x + z)^3;\quad [0, a] \times [0, b] \times [0, c]$

8. $f(x, y, z) = (x + y - z)^2;\quad [0, a] \times [0, b] \times [0, c]$

In Exercises 9–14, evaluate $\iiint_{\mathcal{W}} f(x, y, z)\,dV$ *for the function* f *and region* $\mathcal{W}$ *specified.*

9. $f(x, y, z) = x + y;\quad \mathcal{W}: y \le z \le x,\quad 0 \le y \le x,\quad 0 \le x \le 1$

10. $f(x, y, z) = e^{x+y+z};\quad \mathcal{W}: 0 \le z \le 1,\quad 0 \le y \le x,\quad 0 \le x \le 1$

11. $f(x, y, z) = xyz;\quad \mathcal{W}: 0 \le z \le 1,\quad 0 \le y \le \sqrt{1-x^2},\quad 0 \le x \le 1$

12. $f(x, y, z) = x;\quad \mathcal{W}: x^2 + y^2 \le z \le 4$

13. $f(x, y, z) = e^z;\quad \mathcal{W}: x + y + z \le 1,\quad x \ge 0,\quad y \ge 0,\quad z \ge 0$

14. $f(x, y, z) = z;\quad \mathcal{W}: x^2 \le y \le 2,\quad 0 \le x \le 1,\quad x - y \le z \le x + y$

15. Calculate the integral of $f(x, y, z) = z$ over the region $\mathcal{W}$ in Figure 11 below the hemisphere of radius 3 and lying over the triangle $\mathcal{D}$ in the xy-plane bounded by $x = 1$, $y = 0$, and $x = y$.

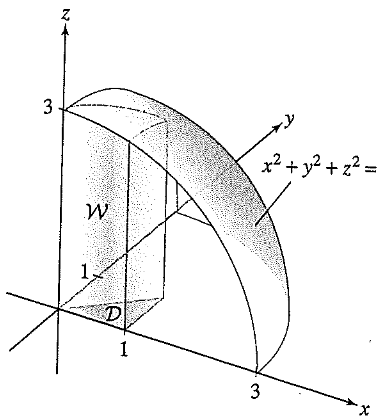


FIGURE 11

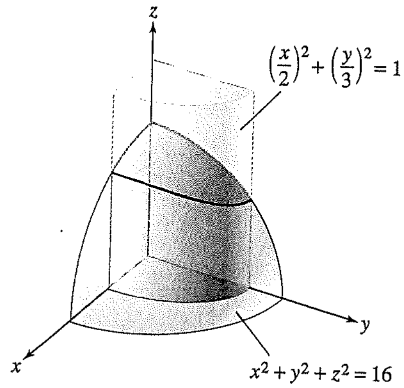


FIGURE 14

16. Calculate the integral of $f(x, y, z) = e^z$ over the tetrahedron $\mathcal{W}$ in Figure 12.

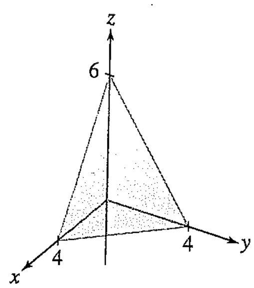


FIGURE 12

17. Integrate $f(x, y, z) = x$ over the region in the first octant ($x \geq 0, y \geq 0, z \geq 0$) above $z = y^2$ and below $z = 8 - 2x^2 - y^2$.

18. Compute the integral of $f(x, y, z) = y^2$ over the region within the cylinder $x^2 + y^2 = 4$, where $0 \leq z \leq y$.

19. Find the triple integral of the function z over the ramp in Figure 13. Here, z is the height above the ground.

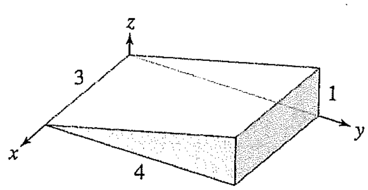


FIGURE 13

20. Find the volume of the solid in $\mathbf{R}^3$ bounded by $y = x^2$, $x = y^2$, $z = x + y + 5$, and $z = 0$.

21. Find the volume of the solid in the octant $x \geq 0, y \geq 0, z \geq 0$ bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

22. Calculate $\iiint_{\mathcal{W}} y\,dV$, where $\mathcal{W}$ is the region above $z = x^2 + y^2$ and below $z = 5$, and bounded by $y = 0$ and $y = 1$.

23. Evaluate $\iiint_{\mathcal{W}} xz\,dV$, where $\mathcal{W}$ is the domain bounded by the elliptic cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the sphere $x^2 + y^2 + z^2 = 16$ in the first octant $x \geq 0, y \geq 0, z \geq 0$ (Figure 14).

24. Describe the domain of integration and evaluate:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy\,dz\,dy\,dx$$

25. Describe the domain of integration of the following integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_1^{\sqrt{5-x^2-z^2}} f(x, y, z)\,dy\,dx\,dz$$

26. Let $\mathcal{W}$ be the region below the paraboloid

$$x^2 + y^2 = z - 2$$

that lies above the part of the plane $x + y + z = 1$ in the first octant ($x \geq 0, y \geq 0, z \geq 0$). Express

$$\iiint_{\mathcal{W}} f(x, y, z)\,dV$$

as an iterated integral (for an arbitrary function f).

27. In Example 5, we expressed a triple integral as an iterated integral in the three orders

$$dz\,dy\,dx, \quad dx\,dz\,dy, \quad \text{and} \quad dy\,dz\,dx$$

Write this integral in the three other orders:

$$dz\,dx\,dy, \quad dx\,dy\,dz, \quad \text{and} \quad dy\,dx\,dz$$

28. Let $\mathcal{W}$ be the region bounded by

$$y + z = 2, \quad 2x = y, \quad x = 0, \quad \text{and} \quad z = 0$$

(Figure 15). Express and evaluate the triple integral of $f(x, y, z) = z$ by projecting $\mathcal{W}$ onto the:

(a) xy-plane. **(b)** yz-plane. **(c)** xz-plane.

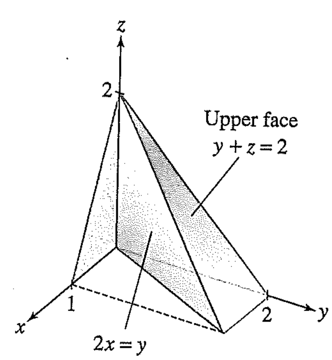


FIGURE 15

4. An ordinary rectangle of sides Δx and Δy has area $\Delta x\,\Delta y$, no matter where it is located in the plane. However, the area of a polar rectangle of sides Δr and $\Delta\theta$ depends on its distance from the origin. How is this difference reflected in the Change of Variables Formula for polar coordinates?

Exercises

In Exercises 1–6, sketch the region $\mathcal{D}$ indicated and integrate $f(x, y)$ over $\mathcal{D}$ using polar coordinates.

1. $f(x, y) = \sqrt{x^2 + y^2}$, $x^2 + y^2 \le 2$

2. $f(x, y) = x^2 + y^2$; $1 \le x^2 + y^2 \le 4$

3. $f(x, y) = xy$; $x \ge 0$, $y \ge 0$, $x^2 + y^2 \le 4$

4. $f(x, y) = y(x^2 + y^2)^3$; $y \ge 0$, $x^2 + y^2 \le 1$

5. $f(x, y) = y(x^2 + y^2)^{-1}$; $y \ge \frac{1}{2}$, $x^2 + y^2 \le 1$

6. $f(x, y) = e^{x^2+y^2}$; $x^2 + y^2 \le R$

In Exercises 7–14, sketch the region of integration and evaluate by changing to polar coordinates.

7. $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)\,dy\,dx$

8. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-y^2}} \sqrt{x^2 + y^2}\,dx\,dy$

9. $\displaystyle\int_{0}^{1/2}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\,dy\,dx$

10. $\displaystyle\int_{0}^{4}\int_{0}^{\sqrt{16-x^2}} \tan^{-1}\frac{y}{x}\,dy\,dx$

11. $\displaystyle\int_{0}^{5}\int_{0}^{y} x\,dx\,dy$

12. $\displaystyle\int_{0}^{2}\int_{x}^{\sqrt{3}x} y\,dy\,dx$

13. $\displaystyle\int_{-1}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)\,dy\,dx$

14. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2 + y^2}}\,dy\,dx$

In Exercises 15–20, calculate the integral over the given region by changing to polar coordinates.

15. $f(x, y) = (x^2 + y^2)^{-2}$; $x^2 + y^2 \le 2$, $x \ge 1$

16. $f(x, y) = x$; $2 \le x^2 + y^2 \le 4$

17. $f(x, y) = |xy|$; $x^2 + y^2 \le 1$

18. $f(x, y) = (x^2 + y^2)^{-3/2}$; $x^2 + y^2 \le 1$, $x + y \ge 1$

19. $f(x, y) = x - y$; $x^2 + y^2 \le 1$, $x + y \ge 1$

20. $f(x, y) = y$; $x^2 + y^2 \le 1$, $(x - 1)^2 + y^2 \le 1$

21. Find the volume of the wedge-shaped region (Figure 18) contained in the cylinder $x^2 + y^2 = 9$, bounded above by the plane $z = x$ and below by the xy-plane.

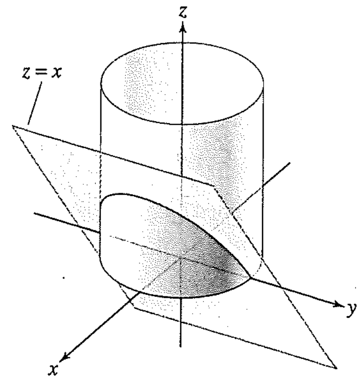


FIGURE 18

22. Let $\mathcal{W}$ be the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$.
(a) Show that the projection of $\mathcal{W}$ on the xy-plane is the disk $x^2 + y^2 \le 2$ (Figure 19).
(b) Compute the volume of $\mathcal{W}$ using polar coordinates.

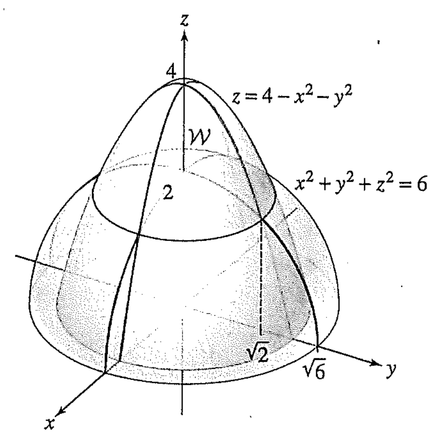


FIGURE 19

23. Evaluate $\displaystyle\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\,dA$, where $\mathcal{D}$ is the domain in Figure 20. *Hint:* Find the equation of the inner circle in polar coordinates and treat the right and left parts of the region separately.

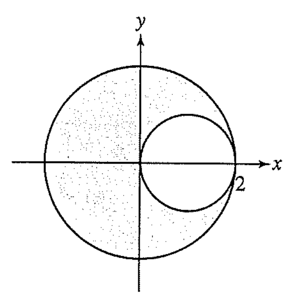


FIGURE 20

24. Evaluate $\displaystyle\iint_{\mathcal{D}} x\sqrt{x^2 + y^2}\,dA$, where $\mathcal{D}$ is the shaded region enclosed by the lemniscate curve $r^2 = \sin 2\theta$ in Figure 21.

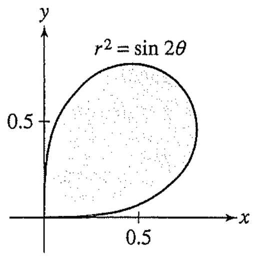


FIGURE 21

25. Let $\mathcal{W}$ be the region above the plane $z = 2$ and below the paraboloid $z = 6 - (x^2 + y^2)$.

(a) Describe $\mathcal{W}$ in cylindrical coordinates.

(b) Use cylindrical coordinates to compute the volume of $\mathcal{W}$.

26. Use cylindrical coordinates to calculate the integral of the function $f(x, y, z) = z$ over the region above the disk $x^2 + y^2 \le 1$ in the xy-plane and below the surface $z = 4 + x^2 + y^2$.

In Exercises 27–32, use cylindrical coordinates to calculate $\iiint_{\mathcal{W}} f(x, y, z)\,dV$ *for the given function and region.*

27. $f(x, y, z) = x^2 + y^2;\quad x^2 + y^2 \le 9,\quad 0 \le z \le 5$

28. $f(x, y, z) = xz;\quad x^2 + y^2 \le 1,\quad x \ge 0,\quad 0 \le z \le 2$

29. $f(x, y, z) = y;\quad x^2 + y^2 \le 1,\quad x \ge 0,\quad y \ge 0,\quad 0 \le z \le 2$

30. $f(x, y, z) = z\sqrt{x^2 + y^2};\quad x^2 + y^2 \le z \le 8 - (x^2 + y^2)$

31. $f(x, y, z) = z;\quad x^2 + y^2 \le z \le 9$

32. $f(x, y, z) = z;\quad 0 \le z \le x^2 + y^2 \le 9$

In Exercises 33–36, express the triple integral in cylindrical coordinates.

33. $\displaystyle\int_{-1}^{1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=0}^{4} f(x, y, z)\,dz\,dy\,dx$

34. $\displaystyle\int_{0}^{1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=0}^{4} f(x, y, z)\,dz\,dy\,dx$

35. $\displaystyle\int_{-1}^{1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=0}^{x^2+y^2} f(x, y, z)\,dz\,dy\,dx$

36. $\displaystyle\int_{0}^{2} \int_{y=0}^{y=\sqrt{2x-x^2}} \int_{z=0}^{\sqrt{x^2+y^2}} f(x, y, z)\,dz\,dy\,dx$

37. Find the equation of the right-circular cone in Figure 22 in cylindrical coordinates and compute its volume.

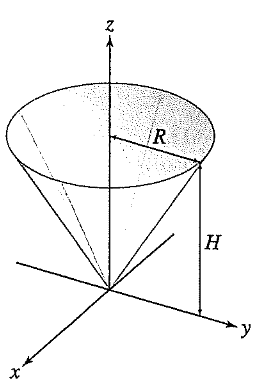


FIGURE 22

38. Use cylindrical coordinates to integrate $f(x, y, z) = z$ over the intersection of the solid hemisphere $x^2 + y^2 + z^2 \le 4$, $z \ge 0$, and the cylinder $x^2 + y^2 =\le 1$.

39. Find the volume of the region appearing between the two surfaces in Figure 23.

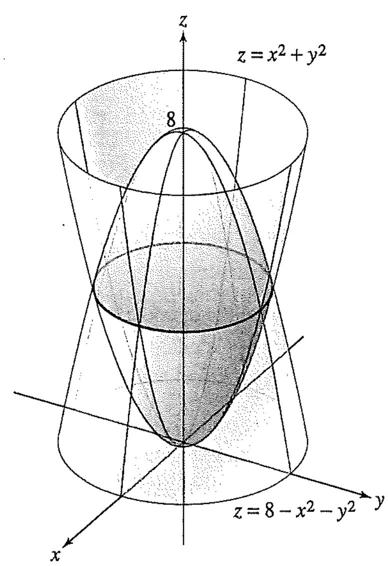


FIGURE 23

40. Use cylindrical coordinates to find the volume of a sphere of radius a from which a central cylinder of radius b has been removed, where $0 < b < a$.

41. Use cylindrical coordinates to show that the volume of a sphere of radius a from which a central cylinder of radius b has been removed, where $0 < b < a$, only depends on the height of the band that results. In particular, this implies that such a band of radius 2 m and height 1 m has the same volume as such a band of radius 6,400 km (the radius of the earth) and height 1 m.

42. Use cylindrical coordinates to find the volume of the region bounded below by the plane $z = 1$ and above by the sphere $x^2 + y^2 + z^2 = 4$.

43. Use spherical coordinates to find the volume of the region bounded below by the plane $z = 1$ and above by the sphere $x^2 + y^2 + z^2 = 4$.

44. Use spherical coordinates to find the volume of a sphere of radius 2 from which a central cylinder of radius 1 has been removed.

In Exercises 45–50, use spherical coordinates to calculate the triple integral of $f(x, y, z)$ *over the given region.*

45. $f(x, y, z) = y;\quad x^2 + y^2 + z^2 \le 1,\quad x, y, z \le 0$

46. $f(x, y, z) = \rho^{-3};\quad 2 \le x^2 + y^2 + z^2 \le 4$

47. $f(x, y, z) = x^2 + y^2;\quad \rho \le 1$

48. $f(x, y, z) = 1;\quad x^2 + y^2 + z^2 \le 4z,\quad z \ge \sqrt{x^2 + y^2}$

49. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2};\quad x^2 + y^2 + z^2 \le 2z$

50. $f(x, y, z) = \rho;\quad x^2 + y^2 + z^2 \le 4,\quad z \le 1,\quad x \ge 0$

51. Use spherical coordinates to evaluate the triple integral of $f(x, y, z) = z$ over the region

$$0 \le \theta \le \frac{\pi}{3}, \qquad 0 \le \phi \le \frac{\pi}{2}, \qquad 1 \le \rho \le 2$$

52. Find the volume of the region lying above the cone $\phi = \phi_0$ and below the sphere $\rho = R$.

53. Calculate the integral of

$$f(x, y, z) = z(x^2 + y^2 + z^2)^{-3/2}$$

over the part of the ball $x^2 + y^2 + z^2 \le 16$ defined by $z \ge 2$.

54. Calculate the volume of the cone in Figure 22, using spherical coordinates.

55. Calculate the volume of the sphere $x^2 + y^2 + z^2 = a^2$, using both spherical and cylindrical coordinates.

56. Let $\mathcal{W}$ be the region within the cylinder $x^2 + y^2 = 2$ between $z = 0$ and the cone $z = \sqrt{x^2 + y^2}$. Calculate the integral of $f(x, y, z) = x^2 + y^2$ over $\mathcal{W}$, using both spherical and cylindrical coordinates.

57. Bell-Shaped Curve One of the key results in calculus is the computation of the area under the bell-shaped curve (Figure 24):

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

This integral appears throughout engineering, physics, and statistics, and although e^{-x^2} does not have an elementary antiderivative, we can compute I using multiple integration.

(a) Show that $I^2 = J$, where J is the improper double integral

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

Hint: Use Fubini's Theorem and $e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$.

(b) Evaluate J in polar coordinates.

(c) Prove that $I = \sqrt{\pi}$.

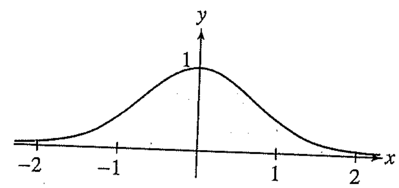


FIGURE 24 The bell-shaped curve $y = e^{-x^2}$.

Further Insights and Challenges

58. An Improper Multiple Integral Show that a triple integral of $(x^2 + y^2 + z^2 + 1)^{-2}$ over all of $\mathbf{R}^3$ is equal to π^2. This is an improper integral, so integrate first over $\rho \le R$ and let $R \to \infty$.

59. Prove the formula

$$\iint_{\mathcal{D}} \ln r\, dA = -\frac{\pi}{2}$$

where $r = \sqrt{x^2 + y^2}$ and $\mathcal{D}$ is the unit disk $x^2 + y^2 \le 1$. This is an improper integral since $\ln r$ is not defined at $(0, 0)$, so integrate first over the annulus $a \le r \le 1$, where $0 < a < 1$, and let $a \to 0$.

60. Recall that the improper integral $\int_0^1 x^{-a}\, dx$ converges if and only if $a < 1$. For which values of a does $\iint_{\mathcal{D}} r^{-a}\, dA$ converge, where $r = \sqrt{x^2 + y^2}$ and $\mathcal{D}$ is the unit disk $x^2 + y^2 \le 1$?

15.5 Applications of Multiple Integrals

This section discusses some applications of multiple integrals. First, we consider quantities (such as mass, charge, and population) that are distributed with a given density δ in $\mathbf{R}^2$ or $\mathbf{R}^3$. In single-variable calculus, we saw that the "total amount" is defined as the integral of density. Similarly, the total amount of a quantity distributed in $\mathbf{R}^2$ or $\mathbf{R}^3$ is defined as the double or triple integral:

$$\text{Total amount} = \iint_{\mathcal{D}} \delta(x, y)\, dA \quad \text{or} \quad \iiint_{\mathcal{W}} \delta(x, y, z)\, dV \qquad (1)$$

The density function δ has units of "amount per unit area" (or per unit volume).

The intuition behind Eq. (1) is similar to that of the single-variable case. Suppose, for example, that $\delta(x, y)$ is population density (Figure 1). When density is constant, the total population is simply density times area:

$$\text{Population} = \text{density (people/km}^2) \times \text{area (km}^2)$$

To treat variable density in the case, say, of a rectangle $\mathcal{R}$, we divide $\mathcal{R}$ into smaller rectangles $\mathcal{R}_{ij}$ of area $\Delta x \Delta y$ on which δ is nearly constant (assuming that δ is continuous on $\mathcal{R}$). The population in $\mathcal{R}_{ij}$ is approximately $\delta(P_{ij})\, \Delta x \Delta y$ for any sample point P_{ij} in

- Radius of gyration: $r_g = (I/M)^{1/2}$
- Random variables X and Y have joint probability density function $p(x, y)$ if

$$P(a \le X \le b; c \le Y \le d) = \int_{x=a}^{b} \int_{y=c}^{d} p(x, y)\, dy\, dx$$

- A joint probability density function must satisfy $p(x, y) \ge 0$ and

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} p(x, y)\, dy\, dx = 1$$

15.5 EXERCISES

Preliminary Questions

1. What is the mass density $\delta(x, y, z)$ of a solid of volume 5 m^3 with uniform mass density and total mass 25 kg?

2. A domain $\mathcal{D}$ in $\mathbf{R}^2$ with uniform mass density is symmetric with respect to the y-axis. Which of the following are true?

(a) $x_{CM} = 0$ (b) $y_{CM} = 0$ (c) $I_x = 0$ (d) $I_y = 0$

3. If $p(x, y)$ is the joint probability density function of random variables X and Y, what does the double integral of $p(x, y)$ over $[0, 1] \times [0, 1]$ represent? What does the integral of $p(x, y)$ over the triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ represent?

Exercises

1. Find the total mass of the square $0 \le x \le 1$, $0 \le y \le 1$ assuming a mass density of

$$\delta(x, y) = x^2 + y^2$$

2. Calculate the total mass of a plate bounded by $y = 0$ and $y = x^{-1}$ for $1 \le x \le 4$ (in meters) assuming a mass density of $\delta(x, y) = y/x$ kg/m^2.

3. Find the total charge in the region under the graph of $y = 4e^{-x^2/2}$ for $0 \le x \le 10$ (in centimeters) assuming a charge density of $\delta(x, y) = 10^{-6}xy$ coulombs per square centimeter (C/cm^2).

4. Find the total population within a 4-km radius of the city center (located at the origin) assuming a population density of $\delta(x, y) = 2000(x^2 + y^2)^{-0.2}$ people per square kilometer.

5. Find the total population within the sector $2|x| \le y \le 8$ assuming a population density of $\delta(x, y) = 100e^{-0.1y}$ people per square kilometer.

6. Find the total mass of the solid region $\mathcal{W}$ defined by $x \ge 0$, $y \ge 0$, $x^2 + y^2 \le 4$, and $x \le z \le 32 - x$ (in centimeters) assuming a mass density of $\delta(x, y, z) = 6y$ g/cm^3.

7. Calculate the total charge of the solid ball $x^2 + y^2 + z^2 \le 5$ (in centimeters) assuming a charge density (in coulombs per cubic centimeter) of

$$\delta(x, y, z) = (3 \cdot 10^{-8})(x^2 + y^2 + z^2)^{1/2}$$

8. Compute the total mass of the plate in Figure 10 assuming a mass density of $f(x, y) = x^2/(x^2 + y^2)$ g/cm^2.

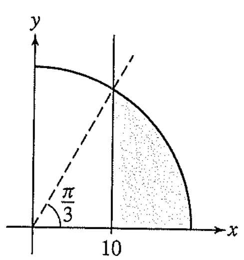


FIGURE 10

9. Assume that the density of the atmosphere as a function of altitude h (in kilometers) above sea level is $\delta(h) = ae^{-bh}$ kg/km^3, where $a = 1.225 \times 10^9$ and $b = 0.13$. Calculate the total mass of the atmosphere contained in the cone-shaped region $\sqrt{x^2 + y^2} \le h \le 3$.

10. Calculate the total charge on a plate $\mathcal{D}$ in the shape of the ellipse with the polar equation

$$r^2 = \left(\frac{1}{6}\sin^2\theta + \frac{1}{9}\cos^2\theta\right)^{-1}$$

with the disk $x^2 + y^2 \le 1$ removed (Figure 11) assuming a charge density of $\rho(r, \theta) = 3r^{-4}$ C/cm^2.

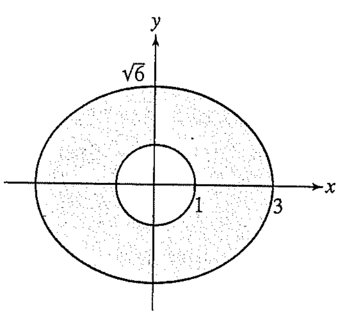


FIGURE 11

In Exercises 11–14, find the centroid of the given region assuming the density $\delta(x, y) = 1$.

11. Region bounded by $y = 1 - x^2$ and $y = 0$

12. Region bounded by $y^2 = x + 4$ and $x = 4$

13. Quarter circle $x^2 + y^2 \le R^2$, $x \ge 0$, $y \ge 0$

14. Infinite lamina bounded by the x- and y-axes and the graph of $y = e^{-x}$

15. CAS Use a computer algebra system to compute numerically the centroid of the shaded region in Figure 12 bounded by $r^2 = \cos 2\theta$ for $x \ge 0$.

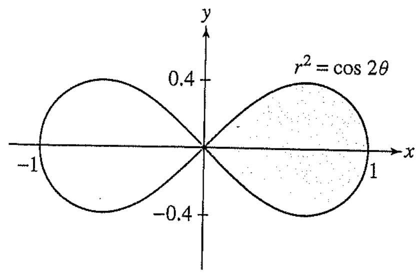


FIGURE 12

16. Show that the centroid of the sector in Figure 13 has y-coordinate

$$\overline{y} = \left(\frac{2R}{3}\right)\left(\frac{\sin\alpha}{\alpha}\right)$$

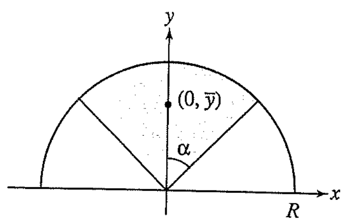


FIGURE 13

In Exercises 17–19, find the centroid of the given solid region assuming a density of $\delta(x, y) = 1$.

17. Hemisphere $x^2 + y^2 + z^2 \le R^2$, $z \ge 0$

18. Region bounded by the xy-plane, the cylinder $x^2 + y^2 = R^2$, and the plane $x/R + z/H = 1$, where $R > 0$ and $H > 0$

19. The "ice cream cone" region $\mathcal{W}$ bounded, in spherical coordinates, by the cone $\phi = \pi/3$ and the sphere $\rho = 2$

20. Show that the z-coordinate of the centroid of the tetrahedron bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

in Figure 14 is $\overline{z} = c/4$. Conclude by symmetry that the centroid is $(a/4, b/4, c/4)$.

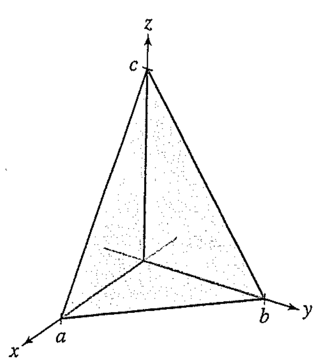


FIGURE 14

21. Find the centroid of the region $\mathcal{W}$ lying above the unit sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$ (Figure 15).

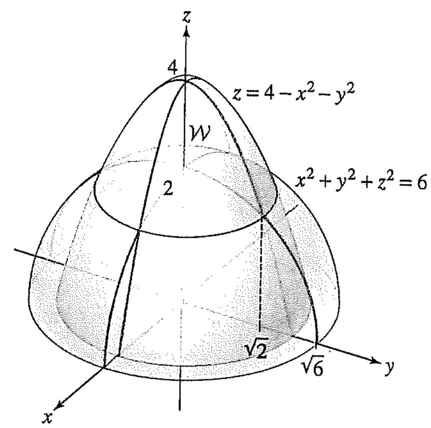


FIGURE 15

22. Let $R > 0$ and $H > 0$, and let $\mathcal{W}$ be the upper half of the ellipsoid $x^2 + y^2 + (Rz/H)^2 = R^2$, where $z \ge 0$ (Figure 16). Find the centroid of $\mathcal{W}$ and show that it depends on the height H but not on the radius R.

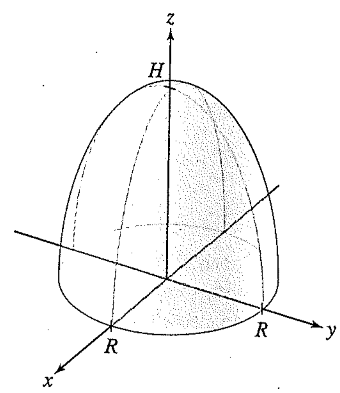


FIGURE 16 Upper half of ellipsoid $x^2 + y^2 + (Rz/H)^2 = R^2$, $z \ge 0$.

In Exercises 23–26, find the center of mass of the region with the given mass density δ.

23. Region bounded by $y = 4 - x$, $x = 0$, $y = 0$; $\delta(x, y) = x$

24. Region bounded by $y^2 = x + 4$ and $x = 0$; $\delta(x, y) = |y|$

25. Region $|x| + |y| \le 1$; $\delta(x, y) = (x + 1)(y + 1)$

26. Semicircle $x^2 + y^2 \le R^2$, $y \ge 0$; $\delta(x, y) = y$

27. Find the z-coordinate of the center of mass of the first octant of the unit sphere with mass density $\delta(x, y, z) = y$ (Figure 17).

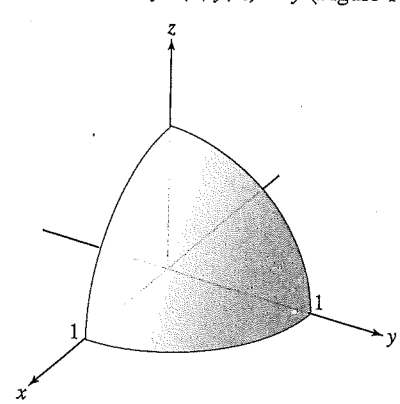


FIGURE 17

28. Find the center of mass of a cylinder of radius 2 and height 4 and mass density e^{-z}, where z is the height above the base.

29. Let $\mathcal{R}$ be the rectangle $[-a, a] \times [b, -b]$ with uniform density and total mass M. Calculate:

(a) The mass density δ of $\mathcal{R}$

(b) I_x and I_0

(c) The radius of gyration about the x-axis

30. Calculate I_x and I_0 for the rectangle in Exercise 29 assuming a mass density of $\delta(x, y) = x$.

31. Calculate I_0 and I_x for the disk $\mathcal{D}$ defined by $x^2 + y^2 \le 16$ (in meters), with total mass 1000 kg and uniform mass density. *Hint:* Calculate I_0 first and observe that $I_0 = 2I_x$. Express your answer in the correct units.

32. Calculate I_x and I_y for the half-disk $x^2 + y^2 \le R^2$, $x \ge 0$ (in meters), with total mass M kilograms and uniform mass density.

In Exercises 33–36, let $\mathcal{D}$ be the triangular domain bounded by the coordinate axes and the line $y = 3 - x$, with mass density $\delta(x, y) = y$. Compute the given quantities.

33. Total mass

34. Center of mass

35. I_x

36. I_0

In Exercises 37–40, let $\mathcal{D}$ be the domain between the line $y = bx/a$ and the parabola $y = bx^2/a^2$, where $a, b > 0$. Assume the mass density is $\delta(x, y) = 1$ for Exercise 37 and $\delta(x, y) = xy$ for Exercises 38–40. Compute the given quantities.

37. Centroid

38. Center of mass

39. I_x

40. I_0

41. Calculate the moment of inertia I_x of the disk $\mathcal{D}$ defined by $x^2 + y^2 \le R^2$ (in meters), with total mass M kg. How much kinetic energy (in joules) is required to rotate the disk about the x-axis with angular velocity 10 rad/s?

42. Calculate the moment of inertia I_z of the box $\mathcal{W} = [-a, a] \times [-a, a] \times [0, H]$ assuming that $\mathcal{W}$ has total mass M.

43. Show that the moment of inertia of a sphere of radius R of total mass M with uniform mass density about any axis passing through the center of the sphere is $\frac{2}{5}MR^2$. Note that the mass density of the sphere is $\delta = M/(\frac{4}{3}\pi R^3)$.

44. Use the result of Exercise 43 to calculate the radius of gyration of a uniform sphere of radius R about any axis through the center of the sphere.

In Exercises 45 and 46, prove the formula for the right circular cylinder in Figure 18.

45. $I_z = \frac{1}{2}MR^2$

46. $I_x = \frac{1}{4}MR^2 + \frac{1}{12}MH^2$

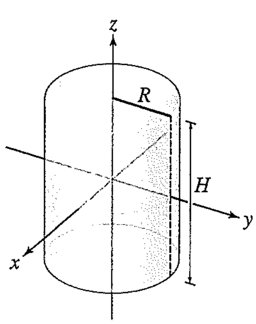


FIGURE 18

47. The yo-yo in Figure 19 is made up of two disks of radius $r = 3$ cm and an axle of radius $b = 1$ cm. Each disk has mass $M_1 = 20$ g, and the axle has mass $M_2 = 5$ g.

(a) Use the result of Exercise 45 to calculate the moment of inertia I of the yo-yo with respect to the axis of symmetry. Note that I is the sum of the moments of the three components of the yo-yo.

(b) The yo-yo is released and falls to the end of a 100-cm string, where it spins with angular velocity ω. The total mass of the yo-yo is $m = 45$ g, so the potential energy lost is $\text{PE} = mgh = (45)(980)100$ g-cm^2/s^2. Find ω using the fact that the potential energy is the sum of the rotational kinetic energy and the translational kinetic energy and that the velocity $v = b\omega$ since the string unravels at this rate.

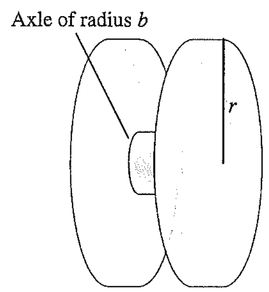


FIGURE 19

48. Calculate I_z for the solid region $\mathcal{W}$ inside the hyperboloid $x^2 + y^2 = z^2 + 1$ between $z = 0$ and $z = 1$.

49. Calculate $P(0 \le X \le 2;\ 1 \le Y \le 2)$, where X and Y have joint probability density function

$$p(x, y) = \begin{cases} \frac{1}{72}(2xy + 2x + y) & \text{if } 0 \le x \le 4 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

50. Calculate the probability that $X + Y \le 2$ for random variables with joint probability density function as in Exercise 49.

51. The lifetime (in months) of two components in a certain device are random variables X and Y that have joint probability distribution function

$$p(x, y) = \begin{cases} \frac{1}{9216}(48 - 2x - y) & \text{if } x \ge 0,\ y \ge 0,\ 2x + y \le 48 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the probability that both components function for at least 12 months without failing. Note that $p(x, y)$ is nonzero only within the triangle bounded by the coordinate axes and the line $2x + y = 48$ shown in Figure 20.

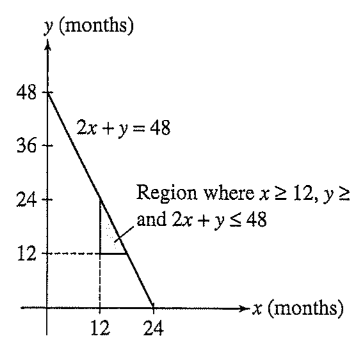


FIGURE 20

52. Find a constant C such that

$$p(x, y) = \begin{cases} Cxy & \text{if } 0 \le x \text{ and } 0 \le y \le 1 - x \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function. Then calculate:

(a) $P(X \le \frac{1}{2};\ Y \le \frac{1}{4})$

(b) $P(X \ge Y)$

15.6 SUMMARY

- Let $G(u, v) = (x(u, v), y(u, v))$ be a mapping. We write $x = x(u, v)$ or $x = x(u, v)$ and, similarly, $y = y(u, v)$ or $y = y(u, v)$. The Jacobian of G is the determinant

$$\text{Jac}(G) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- $\text{Jac}(G) = \text{Jac}(F)^{-1}$, where $F = G^{-1}$.
- Change of Variables Formula: If $G : \mathcal{D}_0 \to \mathcal{D}$ has component functions with continuous partial derivatives and one-to-one on the interior of $\mathcal{D}_0$, and if f is continuous, then

$$\iint_{\mathcal{D}} f(x, y)\, dx\, dy = \iint_{\mathcal{D}_0} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

- The Change of Variables Formula is written symbolically in two and three variables as

$$dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv, \qquad dx\, dy\, dz = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

15.6 EXERCISES

Preliminary Questions

1. Which of the following maps is linear?

(a) (uv, v) **(b)** $(u + v, u)$ **(c)** $(3, e^u)$

2. Suppose that G is a linear map such that $G(2, 0) = (4, 0)$ and $G(0, 3) = (-3, 9)$. Find the images of:

(a) $G(1, 0)$ **(b)** $G(1, 1)$ **(c)** $G(2, 1)$

3. What is the area of $G(\mathcal{R})$ if $\mathcal{R}$ is a rectangle of area 9 and G is a mapping whose Jacobian has constant value 4?

4. Estimate the area of $G(\mathcal{R})$, where $\mathcal{R} = [1, 1.2] \times [3, 3.1]$ and G is a mapping such that $\text{Jac}(G)(1, 3) = 3$.

Exercises

1. Determine the image under $G(u, v) = (2u, u + v)$ of the following sets:

(a) The u- and v-axes

(b) The rectangle $\mathcal{R} = [0, 5] \times [0, 7]$

(c) The line segment joining $(1, 2)$ and $(5, 3)$

(d) The triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$

2. Describe [in the form $y = f(x)$] the images of the lines $u = c$ and $v = c$ under the mapping $G(u, v) = (u/v, u^2 - v^2)$.

3. Let $G(u, v) = (u^2, v)$. Is G one-to-one? If not, determine a domain on which G is one-to-one. Find the image under G of:

(a) The u- and v-axes

(b) The rectangle $\mathcal{R} = [-1, 1] \times [-1, 1]$

(c) The line segment joining $(0, 0)$ and $(1, 1)$

(d) The triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$

4. Let $G(u, v) = (e^u, e^{u+v})$.

(a) Is G one-to-one? What is the image of G?

(b) Describe the images of the vertical lines $u = c$ and the horizontal lines $v = c$.

In Exercises 5–12, let $G(u, v) = (2u + v, 5u + 3v)$ be a map from the uv-plane to the xy-plane.

5. Show that the image of the horizontal line $v = c$ is the line $y = \frac{5}{2}x + \frac{1}{2}c$. What is the image (in slope-intercept form) of the vertical line $u = c$?

6. Describe the image of the line through the points $(u, v) = (1, 1)$ and $(u, v) = (1, -1)$ under G in slope-intercept form.

7. Describe the image of the line $v = 4u$ under G in slope-intercept form.

8. Show that G maps the line $v = mu$ to the line of slope $(5 + 3m)/(2 + m)$ through the origin in the xy-plane.

9. Show that the inverse of G is

$$G^{-1}(x, y) = (3x - y, -5x + 2y)$$

Hint: Show that $G(G^{-1}(x, y)) = (x, y)$ and $G^{-1}(G(u, v)) = (u, v)$.

10. Use the inverse in Exercise 9 to find:

(a) A point in the uv-plane mapping to $(2, 1)$

(b) A segment in the uv-plane mapping to the segment joining $(-2, 1)$ and $(3, 4)$

11. Calculate $\text{Jac}(G) = \dfrac{\partial(x, y)}{\partial(u, v)}$.

12. Calculate $\text{Jac}(G^{-1}) = \dfrac{\partial(u, v)}{\partial(x, y)}$.

In Exercises 13–18, compute the Jacobian (at the point, if indicated).

13. $G(u, v) = (3u + 4v, u - 2v)$

14. $G(r, s) = (rs, r + s)$

15. $G(r, t) = (r \sin t, r - \cos t), \quad (r, t) = (1, \pi)$

16. $G(u, v) = (v \ln u, u^2 v^{-1}), \quad (u, v) = (1, 2)$

17. $G(r, \theta) = (r \cos \theta, r \sin \theta), \quad (r, \theta) = \left(4, \frac{\pi}{6}\right)$

18. $G(u, v) = (ue^v, e^u)$

19. Find a linear mapping G that maps $[0, 1] \times [0, 1]$ to the parallelogram in the xy-plane spanned by the vectors $\langle 2, 3 \rangle$ and $\langle 4, 1 \rangle$.

20. Find a linear mapping G that maps $[0, 1] \times [0, 1]$ to the parallelogram in the xy-plane spanned by the vectors $\langle -2, 5 \rangle$ and $\langle 1, 7 \rangle$.

21. Let $\mathcal{D}$ be the parallelogram in Figure 13. Apply the Change of Variables Formula to the map $G(u, v) = (5u + 3v, u + 4v)$ to evaluate $\iint_{\mathcal{D}} xy \, dx \, dy$ as an integral over $\mathcal{D}_0 = [0, 1] \times [0, 1]$.

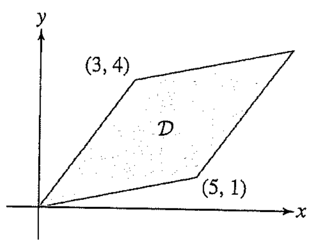


FIGURE 13

22. Let $G(u, v) = (u - uv, uv)$.

(a) Show that the image of the horizontal line $v = c$ is $y = \dfrac{c}{1 - c} x$ if $c \neq 1$, and is the y-axis if $c = 1$.

(b) Determine the images of vertical lines in the uv-plane.

(c) Compute the Jacobian of G.

(d) Observe that by the formula for the area of a triangle, the region $\mathcal{D}$ in Figure 14 has area $\frac{1}{2}(b^2 - a^2)$. Compute this area again, using the Change of Variables Formula applied to G.

(e) Calculate $\iint_{\mathcal{D}} xy \, dx \, dy$.

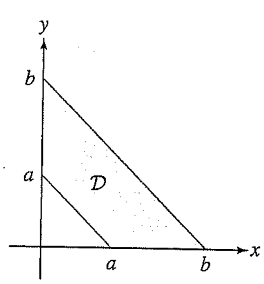


FIGURE 14

23. Let $G(u, v) = (3u + v, u - 2v)$. Use the Jacobian to determine the area of $G(\mathcal{R})$ for:

(a) $\mathcal{R} = [0, 3] \times [0, 5]$ (b) $\mathcal{R} = [2, 5] \times [1, 7]$

24. Find a linear map T that maps $[0, 1] \times [0, 1]$ to the parallelogram $\mathcal{P}$ in the xy-plane with vertices $(0, 0)$, $(2, 2)$, $(1, 4)$, $(3, 6)$. Then calculate the double integral of $e^{2x - y}$ over $\mathcal{P}$ via change of variables.

25. With G as in Example 3, use the Change of Variables Formula to compute the area of the image of $[1, 4] \times [1, 4]$.

In Exercises 26–28, let $\mathcal{R}_0 = [0, 1] \times [0, 1]$ be the unit square. The translate of a map $G_0(u, v) = (\phi(u, v), \psi(u, v))$ is a map

$$G(u, v) = (a + \phi(u, v), b + \psi(u, v))$$

where a, b are constants. Observe that the map G_0 in Figure 15 maps $\mathcal{R}_0$ to the parallelogram $\mathcal{P}_0$ and that the translate

$$G_1(u, v) = (2 + 4u + 2v, 1 + u + 3v)$$

maps $\mathcal{R}_0$ to $\mathcal{P}_1$.

26. Find translates G_2 and G_3 of the mapping G_0 in Figure 15 that map the unit square $\mathcal{R}_0$ to the parallelograms $\mathcal{P}_2$ and $\mathcal{P}_3$.

27. Sketch the parallelogram $\mathcal{P}$ with vertices $(1, 1)$, $(2, 4)$, $(3, 6)$, $(4, 9)$ and find the translate of a linear mapping that maps $\mathcal{R}_0$ to $\mathcal{P}$.

28. Find the translate of a linear mapping that maps $\mathcal{R}_0$ to the parallelogram spanned by the vectors $\langle 3, 9 \rangle$ and $\langle -4, 6 \rangle$ based at $(4, 2)$.

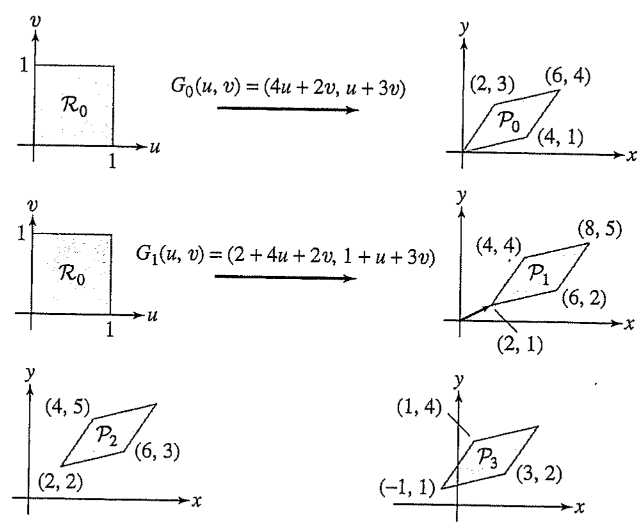


FIGURE 15

29. Let $\mathcal{D} = G(\mathcal{R})$, where $G(u, v) = (u^2, u + v)$ and $\mathcal{R} = [1, 2] \times [0, 6]$. Calculate $\iint_{\mathcal{D}} y \, dx \, dy$. *Note:* It is not necessary to describe $\mathcal{D}$.

30. Let $\mathcal{D}$ be the image of $\mathcal{R} = [1, 4] \times [1, 4]$ under the map $G(u, v) = (u^2/v, v^2/u)$.

(a) Compute $\text{Jac}(G)$.

(b) Sketch $\mathcal{D}$.

(c) Use the Change of Variables Formula to compute $\text{Area}(\mathcal{D})$ and $\iint_{\mathcal{D}} f(x, y) \, dx \, dy$, where $f(x, y) = x + y$.

31. Compute $\iint_{\mathcal{D}} (x + 3y) \, dx \, dy$, where $\mathcal{D}$ is the shaded region in Figure 16. *Hint:* Use the map $G(u, v) = (u - 2v, v)$.

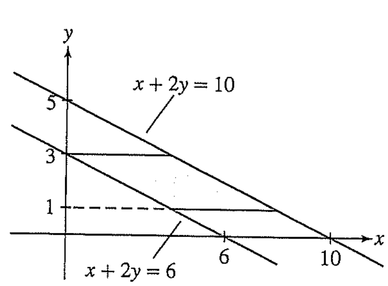


FIGURE 16