- The graph of a continuous real-valued function $f(x, y)$ is the surface in $\mathbf{R}^3$ consisting of the points $(a, b, f(a, b))$ for (a, b) in the domain $\mathcal{D}$ of f.
- A *vertical trace* is a curve obtained by intersecting the graph with a vertical plane $x = a$ or $y = b$.
- A *level curve* is a curve in the xy-plane defined by an equation $f(x, y) = c$. The level curve $f(x, y) = c$ is the projection onto the xy-plane of the horizontal trace curve, obtained by intersecting the graph with the horizontal plane $z = c$.
- A *contour map* shows the level curves $f(x, y) = c$ for equally spaced values of c. The spacing m is called the *contour interval*.
- When reading a contour map, keep in mind:
  - Your altitude does not change when you hike along a level curve.
  - Your altitude increases or decreases by m (the contour interval) when you hike from one level curve to the next.
- The spacing of the level curves indicates steepness: They are closer together where the graph is steeper.
- The *average rate of change* from P to Q is the ratio $\dfrac{\Delta\text{altitude}}{\Delta\text{horizontal}}$.
- A direction of steepest ascent at a point P is a direction along which $f(x, y)$ increases most rapidly. The steepest direction is obtained (approximately) by drawing the segment from P to the nearest point on the next level curve.
- Level surfaces can be used to understand a function $f(x, y, z)$. In the case where the function represents temperature, we call the level surfaces isotherms.

14.1 EXERCISES

Preliminary Questions

1. What is the difference between a horizontal trace and a level curve? How are they related?

2. Describe the trace of $f(x, y) = x^2 - \sin(x^3 y)$ in the xz-plane.

3. Is it possible for two different level curves of a function to intersect? Explain.

4. Describe the contour map of $f(x, y) = x$ with contour interval 1.

5. How will the contour maps of

$$f(x, y) = x \quad \text{and} \quad g(x, y) = 2x$$

with contour interval 1 look different?

Exercises

In Exercises 1–4, evaluate the function at the specified points.

1. $f(x, y) = x + yx^3$, $(2, 2), (-1, 4)$

2. $g(x, y) = \dfrac{y}{x^2 + y^2}$, $(1, 3), (3, -2)$

3. $h(x, y, z) = xyz^{-2}$, $(3, 8, 2), (3, -2, -6)$

4. $Q(y, z) = y^2 + y \sin z$, $(y, z) = \left(2, \frac{\pi}{2}\right), \left(-2, \frac{\pi}{6}\right)$

In Exercises 5–12, sketch the domain of the function.

5. $f(x, y) = 12x - 5y$

6. $f(x, y) = \sqrt{81 - x^2}$

7. $f(x, y) = \ln(4x^2 - y)$

8. $h(x, t) = \dfrac{1}{x + t}$

9. $g(y, z) = \dfrac{1}{z + y^2}$

10. $f(x, y) = \sin \dfrac{y}{x}$

11. $F(I, R) = \sqrt{IR}$

12. $f(x, y) = \cos^{-1}(x + y)$

In Exercises 13–16, describe the domain and range of the function.

13. $f(x, y, z) = xz + e^y$

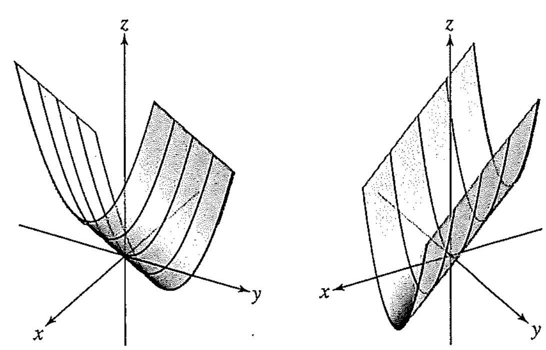
14. $f(x, y, z) = x\sqrt{y + z}e^{z/x}$

15. $P(r, s, t) = \sqrt{16 - r^2 s^2 t^2}$

16. $g(r, s) = \cos^{-1}(rs)$

17. Match graphs (A) and (B) in Figure 20 with the functions:

(i) $f(x, y) = -x + y^2$ (ii) $g(x, y) = x + y^2$



(A) (B)

FIGURE 20

18. Match each of graphs (A) and (B) in Figure 21 with one of the following functions:

(i) $f(x, y) = (\cos x)(\cos y)$

(ii) $g(x, y) = \cos(x^2 + y^2)$

22. insc:

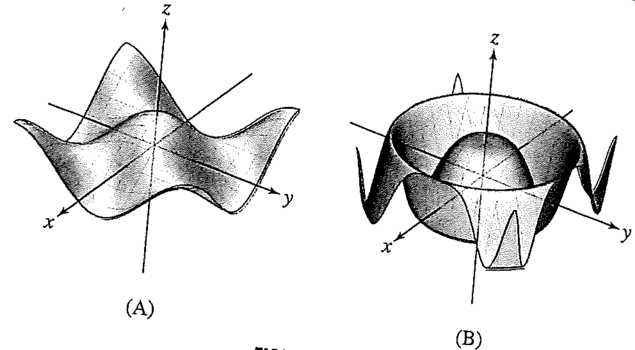


FIGURE 21

19. Match the functions (a)–(f) with their graphs (A)–(F) in Figure 22.

(a) $f(x, y) = |x| + |y|$

(b) $f(x, y) = \cos(x - y)$

(c) $f(x, y) = \dfrac{-1}{1 + 9x^2 + y^2}$

(d) $f(x, y) = \cos(y^2)e^{-0.1(x^2+y^2)}$

(e) $f(x, y) = \dfrac{-1}{1 + 9x^2 + 9y^2}$

(f) $f(x, y) = \cos(x^2 + y^2)e^{-0.1(x^2+y^2)}$

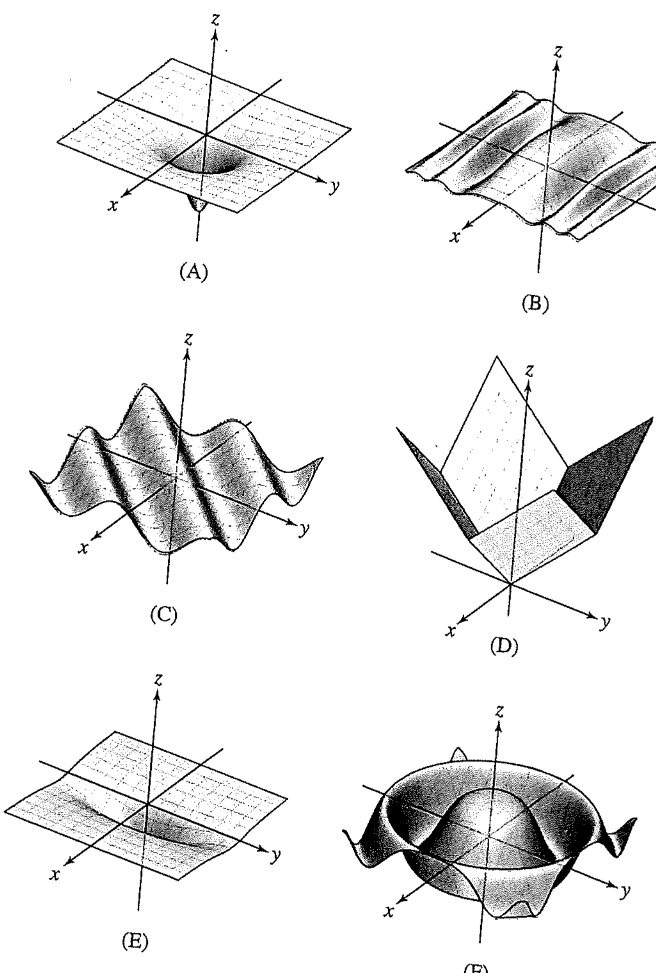


FIGURE 22

20. Match the functions (a)–(d) with their contour maps (A)–(D) in Figure 23.

(a) $f(x, y) = 3x + 4y$

(b) $g(x, y) = x^3 - y$

(c) $h(x, y) = 4x - 3y$

(d) $k(x, y) = x^2 - y$

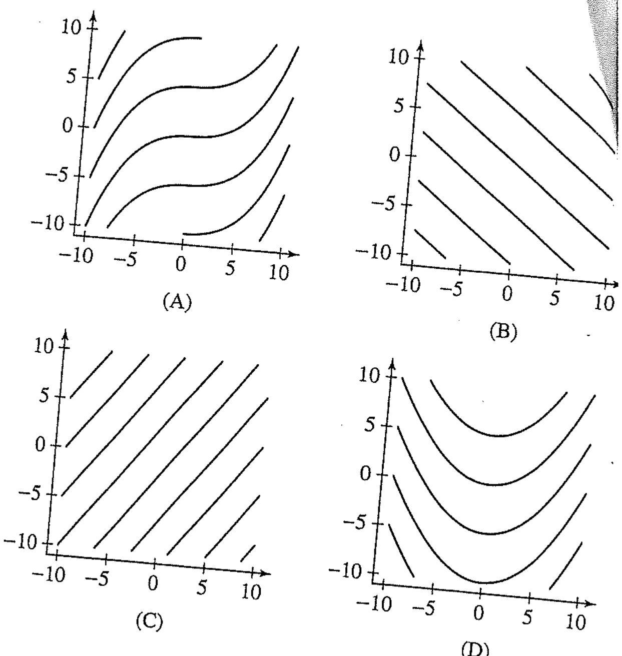


FIGURE 23

In Exercises 21–26, sketch the graph and draw several vertical and horizontal traces.

21. $f(x, y) = 12 - 3x - 4y$

22. $f(x, y) = \sqrt{4 - x^2 - y^2}$

23. $f(x, y) = x^2 + 4y^2$

24. $f(x, y) = y^2$

25. $f(x, y) = \sin(x - y)$

26. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

27. Sketch contour maps of $f(x, y) = x + y$ with contour intervals $m = 1$ and 2.

28. Sketch the contour map of $f(x, y) = x^2 + y^2$ with level curves $c = 0, 4, 8, 12, 16$.

In Exercises 29–36, draw a contour map of $f(x, y)$ with an appropriate contour interval, showing at least six level curves.

29. $f(x, y) = x^2 - y$

30. $f(x, y) = \dfrac{y}{x^2}$

31. $f(x, y) = \dfrac{y}{x}$

32. $f(x, y) = xy$

33. $f(x, y) = x^2 + 4y^2$

34. $f(x, y) = x + 2y - 1$

35. $f(x, y) = x^2$

36. $f(x, y) = 3x^2 - y^2$

37. Find the linear function whose contour map (with contour interval $m = 6$) is shown in Figure 24. What is the linear function if $m = 3$ (and the curve labeled $c = 6$ is relabeled $c = 3$)?

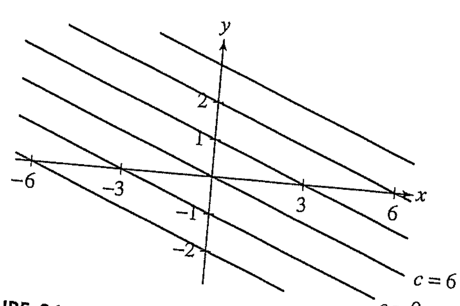


FIGURE 24 Contour map with contour interval $m = 6$.

38. Use the contour map in Figure 25 to calculate the average rate of change:
(a) from A to B. (b) from A to C.

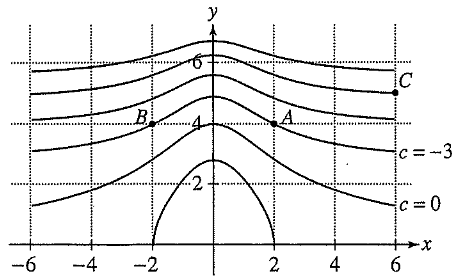


FIGURE 25

39. Referring to Figure 26, answer the following questions:
(a) At which of (A)–(C) is pressure increasing in the northern direction?
(b) At which of (A)–(C) is pressure increasing in the easterly direction?
(c) In which direction at (B) is pressure increasing most rapidly?

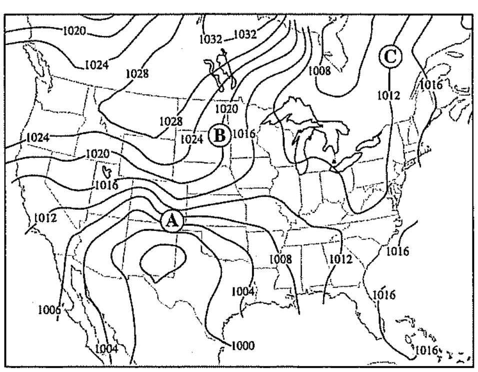


FIGURE 26 Atmospheric pressure (in millibars) over North America on March 26, 2009

In Exercises 40–43, let $T(x, y, z)$ denote temperature at each point in space. Draw level surfaces (also called isotherms) corresponding to the fixed temperatures given.

40. $T(x, y, z) = 2x + 3y - z$. $T = 0, 1, 2$

41. $T(x, y, z) = x - y + 2z$. $T = 0, 1, 2$

42. $T(x, y, z) = x^2 + y^2 - z$. $T = 0, 1, 2$

43. $T(x, y, z) = x^2 - y^2 + z^2$. $T = 0, 1, 2, -1, -2$

In Exercises 44–47, $\rho(S, T)$ is seawater density (kilograms per cubic meter) as a function of salinity S (parts per thousand) and temperature T (degrees Celsius). Refer to the contour map in Figure 27.

44. Calculate the average rate of change of ρ with respect to T from B to A.

45. Calculate the average rate of change of ρ with respect to S from B to C.

46. At a fixed level of salinity, is seawater density an increasing or a decreasing function of temperature?

47. Does water density appear to be more sensitive to a change in temperature at point A or point B?

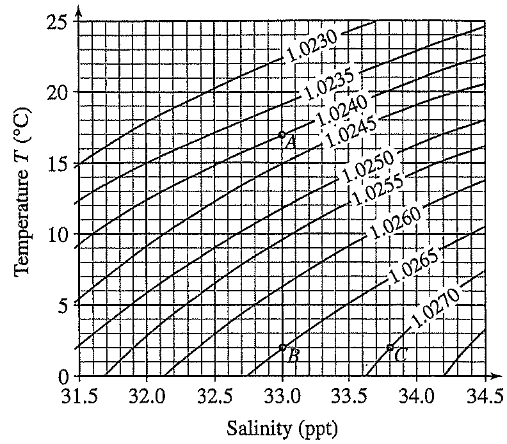


FIGURE 27 Contour map of seawater density $\rho(S, T)$ (kilograms per cubic meter).

In Exercises 48–51, refer to Figure 28.

48. Find the change in elevation from A and B.

49. Estimate the average rate of change from A and B and from A to C.

50. Estimate the average rate of change from A to points i, ii, and iii.

51. Sketch the path of steepest ascent beginning at D.

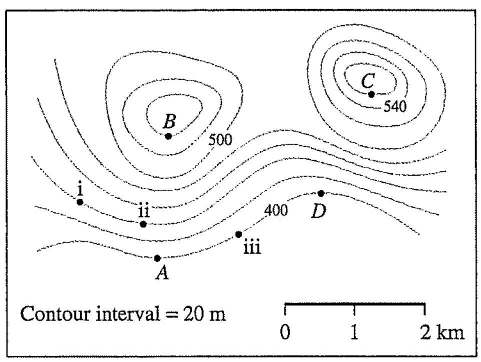


FIGURE 28

52. Let temperature in 3-space be given by $T(x, y, z) = x^2 + y^2 - z$. Draw isotherms corresponding to temperatures $T = -2, -1, 0, 1, 2$.

53. Let temperature in 3-space be given by $T(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2$. Draw isotherms corresponding to temperatures $T = 0, 1, 4$.

54. Let temperature in 3-space be given by $T(x, y, z) = x^2 - y^2 - z$. Draw isotherms corresponding to temperatures $T = -1, 0, 1$.

55. Let temperature in 3-space be given by $T(x, y, z) = x^2 - y^2 - z^2$. Draw isotherms corresponding to temperatures $T = -2, -1, 0, 1, 2$.

In Exercises 9–12, assume that

$$\lim_{(x,y)\to(2,5)} f(x,y) = 3, \qquad \lim_{(x,y)\to(2,5)} g(x,y) = 7$$

to find the limit.

9. $\displaystyle\lim_{(x,y)\to(2,5)} \big(g(x,y) - 2f(x,y)\big)$

10. $\displaystyle\lim_{(x,y)\to(2,5)} f(x,y)^2 g(x,y)$ **11.** $\displaystyle\lim_{(x,y)\to(2,5)} e^{f(x,y)^2 - g(x,y)}$

12. $\displaystyle\lim_{(x,y)\to(2,5)} \frac{f(x,y)}{f(x,y)+g(x,y)}$

13. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$ exist? Explain.

14. Let $f(x,y) = xy/(x^2+y^2)$. Show that $f(x,y)$ approaches zero along the x- and y-axes. Then prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by showing that the limit along the line $y = x$ is nonzero.

15. Let $f(x,y) = \dfrac{x^3+y^3}{xy^2}$. Set $y = mx$ and show that the resulting limit depends on m, and therefore the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

16. Let $f(x,y) = \dfrac{2x^2+3y^2}{xy}$. Set $y = mx$ and show that the resulting limit depends on m, and therefore the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

17. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$$

does not exist by considering the limit along the x-axis.

18. Let $f(x,y) = x^3/(x^2+y^2)$ and $g(x,y) = x^2/(x^2+y^2)$. Using polar coordinates, prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

and that $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. *Hint:* Show that $g(x,y) = \cos^2\theta$ and observe that $\cos\theta$ can take on any value between -1 and 1 as $(x,y) \to (0,0)$.

In Exercises 19–22, use any method to evaluate the limit or show that it does not exist.

19. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$ **20.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

21. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+2y^2}$

22. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4}$

In Exercises 23–24, show that the limit does not exist by approaching the origin along one or more of the coordinate axes.

23. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$ **24.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2+z^2}{x^2+y^2+z^2}$

25. Use the Squeeze Theorem to evaluate

$$\lim_{(x,y)\to(4,0)} (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right)$$

26. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \tan x \sin\left(\frac{1}{|x|+|y|}\right)$.

In Exercises 27–40, evaluate the limit or determine that it does not exist.

27. $\displaystyle\lim_{(z,w)\to(-2,1)} \frac{z^4\cos(\pi w)}{e^{z+w}}$ **28.** $\displaystyle\lim_{(z,w)\to(-1,2)} (z^2w - 9z)$

29. $\displaystyle\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}}$ **30.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1+y^2}$

31. $\displaystyle\lim_{(x,y)\to(3,4)} \frac{1}{\sqrt{x^2+y^2}}$ **32.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

33. $\displaystyle\lim_{(x,y)\to(1,-3)} e^{x-y}\ln(x-y)$ **34.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|}{|x|+|y|}$

35. $\displaystyle\lim_{(x,y)\to(-3,-2)} (x^2y^3 + 4xy)$ **36.** $\displaystyle\lim_{(x,y)\to(2,1)} e^{x^2-y^2}$

37. $\displaystyle\lim_{(x,y)\to(0,0)} \tan(x^2+y^2)\tan^{-1}\left(\frac{1}{x^2+y^2}\right)$

38. $\displaystyle\lim_{(x,y)\to(0,0)} (x+y+2)e^{-1/(x^2+y^2)}$

39. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

40. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|}$

Hint: Rewrite the limit in terms of $u = x-1$ and $v = y-1$.

41. Let $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$.

(a) Show that

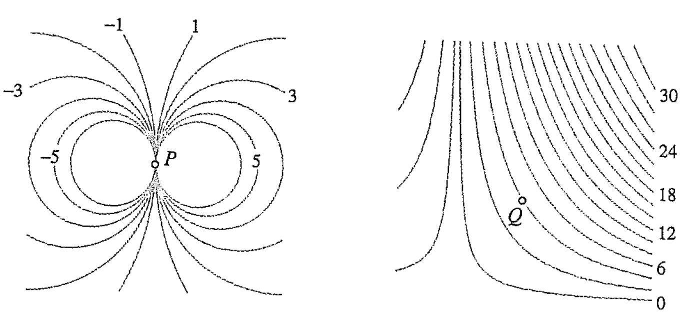
$$|x^3| \le |x|(x^2+y^2), \qquad |y^3| \le |y|(x^2+y^2)$$

(b) Show that $|f(x,y)| \le |x| + |y|$.

(c) Use the Squeeze Theorem to prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

42. Let $a, b \ge 0$. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^ay^b}{x^2+y^2} = 0$ if $a+b > 2$ and that the limit does not exist if $a+b \le 2$.

43. Figure 7 shows the contour maps of two functions. Explain why the limit $\displaystyle\lim_{(x,y)\to P} f(x,y)$ does not exist. Does $\displaystyle\lim_{(x,y)\to Q} g(x,y)$ appear to exist in (B)? If so, what is its limit?



(A) Contour map of $f(x,y)$ (B) Contour map of $g(x,y)$

FIGURE 7

- Compute f_x by holding y constant, and compute f_y by holding x constant.
- $f_x(a, b)$ is the slope at $x = a$ of the tangent line to the trace curve $z = f(x, b)$. Similarly $f_y(a, b)$ is the slope at $y = b$ of the tangent line to the trace curve $z = f(a, y)$.
- For small changes Δx and Δy,

$$f(a + \Delta x, b) - f(a, b) \approx f_x(a, b)\Delta x$$
$$f(a, b + \Delta y) - f(a, b) \approx f_y(a, b)\Delta y$$

More generally, if f is a function of n variables and w is one of the variables, then $\Delta f \approx f_w \Delta w$ if w changes by Δw and all other variables remain fixed.
- The second-order partial derivatives are

$$\frac{\partial^2}{\partial x^2} f = f_{xx}, \qquad \frac{\partial^2}{\partial y\, \partial x} f = f_{xy}, \qquad \frac{\partial^2}{\partial x\, \partial y} f = f_{yx}, \qquad \frac{\partial^2}{\partial y^2} f = f_{yy}$$

- Clairaut's Theorem states that mixed partials are equal—that is, $f_{xy} = f_{yx}$ provided that f_{xy} and f_{yx} are continuous.
- More generally, higher order partial derivatives may be computed in any order. For example, $f_{xyyz} = f_{yxzy}$ if f is a function of x, y, z whose fourth-order partial derivatives are continuous.

14.3 EXERCISES

Preliminary Questions

1. Patricia derived the following *incorrect* formula by misapplying the Product Rule:

$$\frac{\partial}{\partial x}(x^2y^2) = x^2(2y) + y^2(2x)$$

What was her mistake and what is the correct calculation?

2. Explain why it is not necessary to use the Quotient Rule to compute $\frac{\partial}{\partial x}\left(\frac{x+y}{y+1}\right)$. Should the Quotient Rule be used to compute $\frac{\partial}{\partial y}\left(\frac{x+y}{y+1}\right)$?

3. Which of the following partial derivatives should be evaluated without using the Quotient Rule?

(a) $\frac{\partial}{\partial x}\frac{xy}{y^2+1}$ **(b)** $\frac{\partial}{\partial y}\frac{xy}{y^2+1}$ **(c)** $\frac{\partial}{\partial x}\frac{y^2}{y^2+1}$

4. What is f_x, where $f(x, y, z) = (\sin yz)e^{z^3 - z^{-1}\sqrt{y}}$?

5. Assuming the hypotheses of Clairaut's Theorem are satisfied, which of the following partial derivatives are equal to f_{xxy}?

(a) f_{xyx} **(b)** f_{yyx} **(c)** f_{xyy} **(d)** f_{yxx}

Exercises

1. Use the limit definition of the partial derivative to verify the formulas

$$\frac{\partial}{\partial x}xy^2 = y^2, \qquad \frac{\partial}{\partial y}xy^2 = 2xy$$

2. Use the Product Rule to compute $\frac{\partial}{\partial y}(x^2 + y)(x + y^4)$.

3. Use the Quotient Rule to compute $\frac{\partial}{\partial y}\frac{y}{x+y}$.

4. Use the Chain Rule to compute $\frac{\partial}{\partial u}\ln(u^2 + uv)$.

5. Calculate $f_z(2, 3, 1)$, where $f(x, y, z) = xyz$.

6. Explain the relation between the following two formulas (c is a constant):

$$\frac{d}{dx}\sin(cx) = c\cos(cx), \qquad \frac{\partial}{\partial x}\sin(xy) = y\cos(xy)$$

7. The plane $y = 1$ intersects the surface $z = x^4 + 6xy - y^4$ in a certain curve. Find the slope of the tangent line to this curve at the point $P = (1, 1, 6)$.

8. Determine whether the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are positive or negative at the point P on the graph in Figure 7.

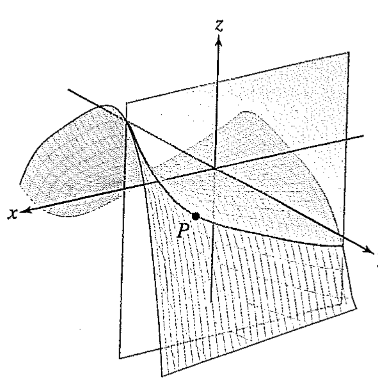


FIGURE 7

In Exercises 9–12, refer to Figure 8.

9. Estimate f_x and f_y at point A.

10. Is f_x positive or negative at B?

11. Starting at point B, in which compass direction (N, NE, SW, etc.) does f increase most rapidly?

12. At which of A, B, or C is f_y smallest?

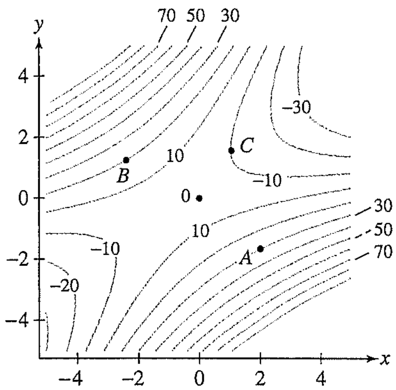


FIGURE 8 Contour map of $f(x, y)$.

In Exercises 13–40, compute the first-order partial derivatives.

13. $z = x^2 + y^2$

14. $z = x^4y^3$

15. $z = x^4y + xy^{-2}$

16. $V = \pi r^2 h$

17. $z = \dfrac{x}{y}$

18. $z = \dfrac{x}{x - y}$

19. $z = \sqrt{9 - x^2 - y^2}$

20. $z = \dfrac{x}{\sqrt{x^2 + y^2}}$

21. $z = (\sin x)(\sin y)$

22. $z = \sin(u^2 v)$

23. $z = \tan \dfrac{x}{y}$

24. $S = \tan^{-1}(wz)$

25. $z = \ln(x^2 + y^2)$

26. $A = \sin(4\theta - 9t)$

27. $W = e^{r+s}$

28. $Q = re^{\theta}$

29. $z = e^{xy}$

30. $R = e^{-v^2/k}$

31. $z = e^{-x^2-y^2}$

32. $P = e^{\sqrt{y^2+z^2}}$

33. $U = \dfrac{e^{-rt}}{r}$

34. $z = y^x$

35. $z = \sinh(x^2 y)$

36. $z = \cosh(t - \cos x)$

37. $w = xy^2z^3$

38. $w = \dfrac{x}{y + z}$

39. $Q = \dfrac{L}{M}e^{-Lt/M}$

40. $w = \dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}$

In Exercises 41–44, compute the given partial derivatives.

41. $f(x, y) = 3x^2y + 4x^3y^2 - 7xy^5$, $f_x(1, 2)$

42. $f(x, y) = \sin(x^2 - y)$, $f_y(0, \pi)$

43. $g(u, v) = u \ln(u + v)$, $g_u(1, 2)$

44. $h(x, z) = e^{xz - x^2z^3}$, $h_z(3, 0)$

Exercises 45 and 46 refer to Example 5.

45. Calculate N for $L = 0.4$, $R = 0.12$, and $d = 10$, and use the linear approximation to estimate ΔN if d is increased from 10 to 10.4.

46. Estimate ΔN if $(L, R, d) = (0.5, 0.15, 8)$ and R is increased from 0.15 to 0.17.

47. The **heat index** I is a measure of how hot it feels when the relative humidity is H (as a percentage) and the actual air temperature is T (in degrees Fahrenheit). An approximate formula for the heat index that is valid for (T, H) near (90, 40) is

$$I(T, H) = 45.33 + 0.6845T + 5.758H - 0.00365T^2 - 0.1565HT + 0.001HT^2$$

(a) Calculate I at $(T, H) = (95, 50)$.

(b) Which partial derivative tells us the increase in I per degree increase in T when $(T, H) = (95, 50)$? Calculate this partial derivative.

48. The **wind-chill temperature** W measures how cold people feel (based on the rate of heat loss from exposed skin) when the outside temperature is T°C (with $T \le 10$) and wind velocity is v m/s (with $v \ge 2$):

$$W = 13.1267 + 0.6215T - 13.947v^{0.16} + 0.486Tv^{0.16}$$

Calculate $\partial W/\partial v$ at $(T, v) = (-10, 15)$ and use this value to estimate ΔW if $\Delta v = 2$.

49. The volume of a right-circular cone of radius r and height h is $V = \frac{\pi}{3}r^2h$. Suppose that $r = h = 12$ cm. What leads to a greater increase in V, a 1-cm increase in r or a 1-cm increase in h? Argue using partial derivatives.

50. Use the linear approximation to estimate the percentage change in volume of a right-circular cone of radius $r = 40$ cm if the height is increased from 40 to 41 cm.

51. Calculate $\partial W/\partial E$ and $\partial W/\partial T$, where $W = e^{-E/kT}$, where k is a constant.

52. Calculate $\partial P/\partial T$ and $\partial P/\partial V$, where pressure P, volume V, and temperature T are related by the Ideal Gas Law, $PV = nRT$ (R and n are constants).

53. Use the contour map of $f(x, y)$ in Figure 9 to explain the following statements:

(a) f_y is larger at P than at Q, and f_x is smaller (more negative) at P than at Q.

(b) $f_x(x, y)$ is decreasing as a function of y; that is, for any fixed value $x = a$, $f_x(a, y)$ is decreasing in y.

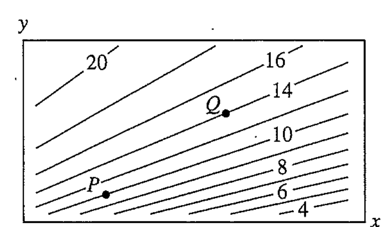


FIGURE 9

54. Estimate the partial derivatives at P of the function whose contour map is shown in Figure 10.

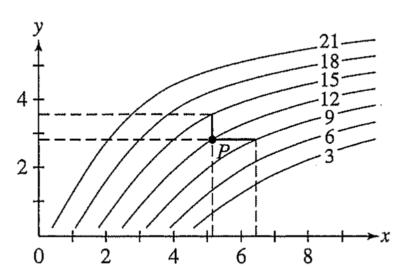


FIGURE 10

55. Over most of the earth, a magnetic compass does not point to true (geographic) north; instead, it points at some angle east or west of true north. The angle D between magnetic north and true north is called the **magnetic declination**. Use Figure 11 to determine which of the following statements is true:

(a) $\left.\frac{\partial D}{\partial y}\right|_A > \left.\frac{\partial D}{\partial y}\right|_B$ **(b)** $\left.\frac{\partial D}{\partial x}\right|_C > 0$ **(c)** $\left.\frac{\partial D}{\partial y}\right|_C > 0$

Note that the horizontal axis increases from right to left because of the way longitude is measured.

Magnetic Declination for the U.S. 2004

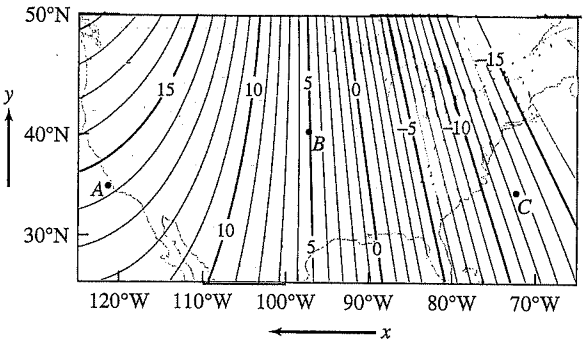


FIGURE 11 Contour interval 1°.

56. Refer to Table 1.

(a) Estimate $\partial \rho/\partial T$ and $\partial \rho/\partial S$ at the points $(S, T) = (34, 2)$ and $(35, 10)$ by computing the average of left-hand and right-hand difference quotients.

(b) For fixed salinity $S = 33$, is ρ concave up or concave down as a function of T? *Hint:* Determine whether the quotients $\Delta\rho/\Delta T$ are increasing or decreasing. What can you conclude about the sign of $\partial^2\rho/\partial T^2$?

TABLE 1 Seawater Density ρ as a Function of Temperature T and Salinity S

T \ S	30	31	32	33	34	35	36
12	22.75	23.51	24.27	25.07	25.82	26.6	27.36
10	23.07	23.85	24.62	25.42	26.17	26.99	27.73
8	23.36	24.15	24.93	25.73	26.5	27.28	29.09
6	23.62	24.44	25.22	26	26.77	27.55	28.35
4	23.85	24.62	25.42	26.23	27	27.8	28.61
2	24	24.78	25.61	26.38	27.18	28.01	28.78
0	24.11	24.92	25.72	26.5	27.34	28.12	28.91

In Exercises 57–62, compute the derivatives indicated.

57. $f(x, y) = 3x^2y - 6xy^4$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$

58. $g(x, y) = \frac{xy}{x - y}$, $\frac{\partial^2 g}{\partial x\,\partial y}$

59. $h(u, v) = \frac{u}{u + 4v}$, $h_{vv}(u, v)$

60. $h(x, y) = \ln(x^3 + y^3)$, $h_{xy}(x, y)$

61. $f(x, y) = x\ln(y^2)$, $f_{yy}(2, 3)$

62. $g(x, y) = xe^{-xy}$, $g_{xy}(-3, 2)$

63. Compute f_{xyxzy} for

$$f(x, y, z) = y\sin(xz)\sin(x + z) + (x + z^2)\tan y + x\tan\left(\frac{z + z^{-1}}{y - y^{-1}}\right)$$

Hint: Use a well-chosen order of differentiation on each term.

64. Let

$$f(x, y, u, v) = \frac{x^2 + e^y v}{3y^2 + \ln(2 + u^2)}$$

What is the fastest way to show that $f_{uvxyvu}(x, y, u, v) = 0$ for all (x, y, u, v)?

In Exercises 65–72, compute the derivative indicated.

65. $f(u, v) = \cos(u + v^2)$, f_{uuv}

66. $g(x, y, z) = x^4y^5z^6$, g_{xxyz}

67. $F(r, s, t) = r(s^2 + t^2)$, F_{rst}

68. $u(x, t) = t^{-1/2}e^{-(x^2/4t)}$, u_{xx}

69. $F(\theta, u, v) = \sinh(uv + \theta^2)$, $F_{uu\theta}$

70. $R(u, v, w) = \frac{u}{v + w}$, R_{uvw}

71. $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, g_{xyz}

72. $u(x, t) = \operatorname{sech}^2(x - t)$, u_{xxx}

73. Find a function such that $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2$.

74. Prove that there does not exist any function $f(x, y)$ such that $\frac{\partial f}{\partial x} = xy$ and $\frac{\partial f}{\partial y} = x^2$. *Hint:* Show that f cannot satisfy Clairaut's Theorem.

75. Assume that f_{xy} and f_{yx} are continuous and that f_{yxx} exists. Show that f_{xyx} also exists and that $f_{yxx} = f_{xyx}$.

76. Show that $u(x, t) = \sin(nx)\,e^{-n^2t}$ satisfies the heat equation for any constant n:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \boxed{3}$$

77. Find all values of A and B such that $f(x, t) = e^{Ax+Bt}$ satisfies Eq. (3).

78. The function

$$f(x, t) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}$$

describes the temperature profile along a metal rod at time $t > 0$ when a burst of heat is applied at the origin (see Example 11). A small bug sitting on the rod at distance x from the origin feels the temperature rise and fall as heat diffuses through the bar. Show that the bug feels the maximum temperature at time $t = \frac{1}{2}x^2$.

*In Exercises 79–82, the **Laplace operator** Δ is defined by $\Delta f = f_{xx} + f_{yy}$. A function $u(x, y)$ satisfying the Laplace equation $\Delta u = 0$ is called **harmonic**.*

79. Show that the following functions are harmonic:

3. Which of (a)–(b) is the linearization of f at (2, 3)?
(a) $L(x, y) = 8 + 5x + 7y$
(b) $L(x, y) = 8 + 5(x - 2) + 7(y - 3)$

4. Estimate $f(2, 3.1)$.

5. Estimate Δf at (2, 3) if $\Delta x = -0.3$ and $\Delta y = 0.2$.

6. Which theorem allows us to conclude that $f(x, y) = x^3y^8$ is differentiable?

Exercises

1. Use Eq. (2) to find an equation of the tangent plane to the graph of $f(x, y) = 2x^2 - 4xy^2$ at $(-1, 2)$.

2. Find the equation of the plane in Figure 10, which is tangent to the graph at $(x, y) = (1, 0.8)$.

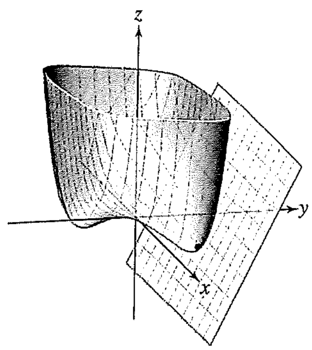


FIGURE 10 Graph of $f(x, y) = 0.2x^4 + y^6 - xy$.

In Exercises 3–10, find an equation of the tangent plane at the given point.

3. $f(x, y) = x^2y + xy^3$, (2, 1)

4. $f(x, y) = \dfrac{x}{\sqrt{y}}$, (4, 4)

5. $f(x, y) = x^2 + y^{-2}$, (4, 1)

6. $G(u, w) = \sin(uw)$, $(\frac{\pi}{6}, 1)$

7. $F(r, s) = r^2s^{-1/2} + s^{-3}$, (2, 1)

8. $g(x, y) = e^{x/y}$, (2, 1)

9. $f(x, y) = \operatorname{sech}(x - y)$, $(\ln 4, \ln 2)$

10. $f(x, y) = \ln(4x^2 - y^2)$, (1, 1)

11. Find the points on the graph of $z = 3x^2 - 4y^2$ at which the vector $\mathbf{n} = \langle 3, 2, 2 \rangle$ is normal to the tangent plane.

12. Find the points on the graph of $z = xy^3 + 8y^{-1}$ where the tangent plane is parallel to $2x + 7y + 2z = 0$.

13. Find the linearization $L(x, y)$ of $f(x, y) = x^2y^3$ at $(a, b) = (2, 1)$. Use it to estimate $f(2.01, 1.02)$ and $f(1.97, 1.01)$, and compare with values obtained using a calculator.

14. Write the linear approximation to $f(x, y) = x(1 + y)^{-1}$ at $(a, b) = (8, 1)$ in the form

$$f(a + h, b + k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$$

Use it to estimate $\dfrac{7.98}{2.02}$ and compare with the value obtained using a calculator.

15. Let $f(x, y) = x^3y^{-4}$. Use Eq. (5) to estimate the change

$$\Delta f = f(2.03, 0.9) - f(2, 1)$$

16. Use the linear approximation to $f(x, y) = \sqrt{x/y}$ at (9, 4) to estimate $\sqrt{9.1/3.9}$.

17. Use the linear approximation of $f(x, y) = e^{x^2+y}$ at (0, 0) to estimate $f(0.01, -0.02)$. Compare with the value obtained using a calculator.

18. Let $f(x, y) = x^2/(y^2 + 1)$. Use the linear approximation at an appropriate point (a, b) to estimate $f(4.01, 0.98)$.

19. Find the linearization of $f(x, y, z) = z\sqrt{x + y}$ at (8, 4, 5).

20. Find the linearization to $f(x, y, z) = xy/z$ at the point (2, 1, 2). Use it to estimate $f(2.05, 0.9, 2.01)$ and compare with the value obtained from a calculator.

21. Estimate $f(2.1, 3.8)$ assuming that

$$f(2, 4) = 5, \quad f_x(2, 4) = 0.3, \quad f_y(2, 4) = -0.2$$

22. Estimate $f(1.02, 0.01, -0.03)$ assuming that

$$f(1, 0, 0) = -3, \quad f_x(1, 0, 0) = -2$$
$$f_y(1, 0, 0) = 4, \quad f_z(1, 0, 0) = 2$$

In Exercises 23–28, use the linear approximation to estimate the value. Compare with the value given by a calculator.

23. $(2.01)^3(1.02)^2$

24. $\dfrac{4.1}{7.9}$

25. $\sqrt{3.01^2 + 3.99^2}$

26. $\dfrac{0.98^2}{2.01^3 + 1}$

27. $\sqrt{(1.9)(2.02)(4.05)}$

28. $\dfrac{8.01}{\sqrt{(1.99)(2.01)}}$

29. Suppose that the plane tangent to $z = f(x, y)$ at $(-2, 3, 4)$ has equation $4x + 2y + z = 2$. Estimate $f(-2.1, 3.1)$.

30. In the derivation of the equation for the tangent plane that appears in Theorem 1, we chose $\mathbf{n} = \mathbf{v} \times \mathbf{u}$ for the normal vector to the plane. How would the choice of $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ for the normal vector have affected the resultant equation?

In Exercises 31–34, let $I = W/H^2$ denote the BMI described in Example 5.

31. A boy has weight $W = 34$ kg and height $H = 1.3$ m. Use the linear approximation to estimate the change in I if (W, H) changes to (36, 1.32).

32. Suppose that $(W, H) = (34, 1.3)$. Use the linear approximation to estimate the increase in H required to keep I constant if W increases to 35.

33. (a) Show that $\Delta I \approx 0$ if $\Delta H/\Delta W \approx H/2W$.
(b) Suppose that $(W, H) = (25, 1.1)$. What increase in H will leave I (approximately) constant if W is increased by 1 kg?

34. Estimate the change in height that will decrease I by 1 if $(W, H) = (25, 1.1)$, assuming that W remains constant.

Exercises

1. Let $f(x, y) = xy^2$ and $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, t^3 \right\rangle$.

(a) Calculate ∇f and $\mathbf{r}'(t)$.

(b) Use the Chain Rule for Paths to evaluate $\frac{d}{dt} f(\mathbf{r}(t))$ at $t = 1$ and $t = -1$.

2. Let $f(x, y) = e^{xy}$ and $\mathbf{r}(t) = \left\langle t^3, 1 + t \right\rangle$.

(a) Calculate ∇f and $\mathbf{r}'(t)$.

(b) Use the Chain Rule for Paths to calculate $\frac{d}{dt} f(\mathbf{r}(t))$.

(c) Write out the composite $f(\mathbf{r}(t))$ as a function of t and differentiate. Check that the result agrees with part (b).

3. Figure 15 shows the level curves of a function $f(x, y)$ and a path $\mathbf{r}(t)$, traversed in the direction indicated. State whether the derivative $\frac{d}{dt} f(\mathbf{r}(t))$ is positive, negative, or zero at points A–D.

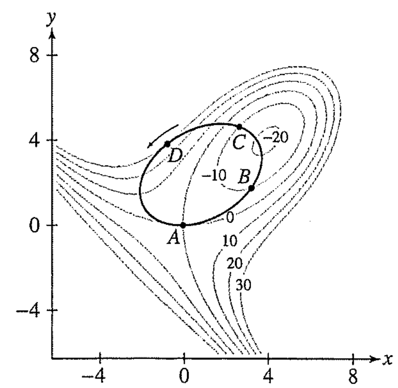


FIGURE 15

4. Let $f(x, y) = x^2 + y^2$ and $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$.

(a) Find $\frac{d}{dt} f(\mathbf{r}(t))$ without making any calculations. Explain.

(b) Verify your answer to (a) using the Chain Rule.

In Exercises 5–8, calculate the gradient.

5. $f(x, y) = \cos(x^2 + y)$

6. $g(x, y) = \dfrac{x}{x^2 + y^2}$

7. $h(x, y, z) = xyz^{-3}$

8. $r(x, y, z, w) = xze^{yw}$

In Exercises 9–20, use the Chain Rule to calculate $\frac{d}{dt} f(\mathbf{r}(t))$.

9. $f(x, y) = 3x - 7y, \quad \mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad t = 0$

10. $f(x, y) = 3x - 7y, \quad \mathbf{r}(t) = \left\langle t^2, t^3 \right\rangle, \quad t = 2$

11. $f(x, y) = x^2 - 3xy, \quad \mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad t = 0$

12. $f(x, y) = x^2 - 3xy, \quad \mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \frac{\pi}{2}$

13. $f(x, y) = \sin(xy), \quad \mathbf{r}(t) = \left\langle e^{2t}, e^{3t} \right\rangle, \quad t = 0$

14. $f(x, y) = \cos(y - x), \quad \mathbf{r}(t) = \left\langle e^t, e^{2t} \right\rangle, \quad t = \ln 3$

15. $f(x, y) = x - xy, \quad \mathbf{r}(t) = \left\langle t^2, t^2 - 4t \right\rangle, \quad t = 4$

16. $f(x, y) = xe^y, \quad \mathbf{r}(t) = \left\langle t^2, t^2 - 4t \right\rangle, \quad t = 0$

17. $f(x, y) = \ln x + \ln y, \quad \mathbf{r}(t) = \left\langle \cos t, t^2 \right\rangle, \quad t = \frac{\pi}{4}$

18. $g(x, y, z) = xye^z, \quad \mathbf{r}(t) = \left\langle t^2, t^3, t - 1 \right\rangle, \quad t = 1$

19. $g(x, y, z) = xyz^{-1}, \quad \mathbf{r}(t) = \left\langle e^t, t, t^2 \right\rangle, \quad t = 1$

20. $g(x, y, z, w) = x + 2y + 3z + 5w, \quad \mathbf{r}(t) = \left\langle t^2, t^3, t, t-2 \right\rangle, \quad t = 1$

In Exercises 21–30, calculate the directional derivative in the direction of $\mathbf{v}$ *at the given point. Remember to normalize the direction vector.*

21. $f(x, y) = x^2 + y^3, \quad \mathbf{v} = \langle 4, 3 \rangle, \quad P = (1, 2)$

22. $f(x, y) = x^2y^3, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}, \quad P = (-2, 1)$

23. $f(x, y) = x^2y^3, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}, \quad P = \left(\frac{1}{6}, 3\right)$

24. $f(x, y) = \sin(x - y), \quad \mathbf{v} = \langle 1, 1 \rangle, \quad P = \left(\frac{\pi}{2}, \frac{\pi}{6}\right)$

25. $f(x, y) = \tan^{-1}(xy), \quad \mathbf{v} = \langle 1, 1 \rangle, \quad P = (3, 4)$

26. $f(x, y) = e^{xy - y^2}, \quad \mathbf{v} = \langle 12, -5 \rangle, \quad P = (2, 2)$

27. $f(x, y) = \ln(x^2 + y^2), \quad \mathbf{v} = 3\mathbf{i} - 2\mathbf{j}, \quad P = (1, 0)$

28. $g(x, y, z) = z^2 - xy^2, \quad \mathbf{v} = \langle -1, 2, 2 \rangle, \quad P = (2, 1, 3)$

29. $g(x, y, z) = xe^{-yz}, \quad \mathbf{v} = \langle 1, 1, 1 \rangle, \quad P = (1, 2, 0)$

30. $g(x, y, z) = x\ln(y + z), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad P = (2, e, e)$

31. Find the directional derivative of $f(x, y) = x^2 + 4y^2$ at the point $P = (3, 2)$ in the direction pointing to the origin.

32. Find the directional derivative of $f(x, y, z) = xy + z^3$ at the point $P = (3, -2, -1)$ in the direction pointing to the origin.

33. A bug located at $(3, 9, 4)$ begins walking in a straight line toward $(5, 7, 3)$. At what rate is the bug's temperature changing if the temperature is $T(x, y, z) = xe^{y-z}$? Units are in meters and degrees Celsius.

In Exercises 34–35, assume that the positive x-axis points East and the positive y-axis points North.

34. Suppose you are hiking on a terrain modeled by $z = x^2 + y^2 - y$. You are at the point $(1, 2, 3)$.

(a) Determine the slope you would encounter if you headed due East from your position. What angle of inclination does that correspond to?

(b) Determine the slope you would encounter if you headed due North from your position. What angle of inclination does that correspond to?

(c) Determine the slope you would encounter if you headed due North-East from your position. What angle of inclination does that correspond to?

(d) Determine the steepest slope you could encounter from your position.

35. Suppose you are hiking on a terrain modeled by $z = xy + y^3 - x^2$. You are at the point $(2, 1, -1)$.

(a) Determine the slope you would encounter if you headed due West from your position. What angle of inclination does that correspond to?

(b) Determine the slope you would encounter if you headed due North-West from your position. What angle of inclination does that correspond to?

(c) Determine the slope you would encounter if you headed due South-East from your position. What angle of inclination does that correspond to?

(d) Determine the steepest slope you could encounter from your position, and the compass direction measured in degrees from East that you would head to realize this steepest slope.

36. The temperature at location (x, y) is $T(x, y) = 20 + 0.1(x^2 - xy)$ (degrees Celsius). Beginning at $(200, 0)$ at time $t = 0$ (seconds), a bug travels counterclockwise along a circle of radius 200 cm centered at the origin, at a speed of 3 cm/s. How fast is the temperature changing when the bug is at a position on the circle corresponding to an angle of $\theta = \pi/3$?

37. Suppose that $\nabla f_P = \langle 2, -4, 4\rangle$. Is f increasing or decreasing at P in the direction $\mathbf{v} = \langle 2, 1, 3\rangle$?

38. Let $f(x, y) = xe^{x^2 - y}$ and $P = (1, 1)$.

(a) Calculate $\|\nabla f_P\|$.

(b) Find the rate of change of f in the direction ∇f_P.

(c) Find the rate of change of f in the direction of a vector making an angle of 45° with ∇f_P.

39. Let $f(x, y, z) = \sin(xy + z)$ and $P = (0, -1, \pi)$. Calculate $D_{\mathbf{u}} f(P)$, where $\mathbf{u}$ is a unit vector making an angle $\theta = 30°$ with ∇f_P.

40. Let $T(x, y)$ be the temperature at location (x, y) on a thin sheet of metal. Assume that $\nabla T = \langle y - 4, x + 2y\rangle$. Let $\mathbf{r}(t) = \langle t^2, t\rangle$ be a path on the sheet. Find the values of t such that

$$\frac{d}{dt} T(\mathbf{r}(t)) = 0$$

41. Find a vector normal to the surface $x^2 + y^2 - z^2 = 6$ at $P = (3, 1, 2)$.

42. Find a vector normal to the surface $3z^3 + x^2 y - y^2 x = 1$ at $P = (1, -1, 1)$.

43. Find the two points on the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

where the tangent plane is normal to $\mathbf{v} = \langle 1, 1, -2\rangle$.

In Exercises 44–47, find an equation of the tangent plane to the surface at the given point.

44. $x^2 + 3y^2 + 4z^2 = 20, \quad P = (2, 2, 1)$

45. $xz + 2x^2 y + y^2 z^3 = 11, \quad P = (2, 1, 1)$

46. $x^2 + z^2 e^{y-x} = 13, \quad P = \left(2, 3, \frac{3}{\sqrt{e}}\right)$

47. $\ln[1 + 4x^2 + 9y^4] - 0.1z^2 = 0, \quad P = (3, 1, 6.1876)$

48. Verify what is clear from Figure 16: Every tangent plane to the cone $x^2 + y^2 - z^2 = 0$ passes through the origin.

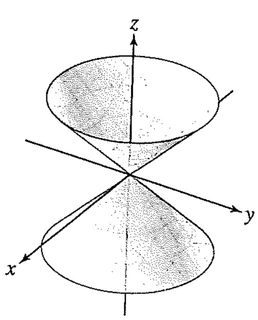


FIGURE 16 Graph of $x^2 + y^2 - z^2 = 0$.

49. CAS Use a computer algebra system to produce a contour of $f(x, y) = x^2 - 3xy + y - y^2$ together with its gradient vector field on the domain $[-4, 4] \times [-4, 4]$.

50. Find a function $f(x, y, z)$ such that ∇f is the constant vector $\langle 1, 3, 1\rangle$.

51. Find a function $f(x, y, z)$ such that $\nabla f = \langle 2x, 1, 2\rangle$.

52. Find a function $f(x, y, z)$ such that $\nabla f = \langle x, y^2, z^3\rangle$.

53. Find a function $f(x, y, z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

54. Find a function $f(x, y)$ such that $\nabla f = \langle y, x\rangle$.

55. Show that there does not exist a function $f(x, y)$ such that $\nabla f = \langle y^2, x\rangle$. *Hint:* Use Clairaut's Theorem $f_{xy} = f_{yx}$.

56. Let $\Delta f = f(a + h, b + k) - f(a, b)$ be the change in f at $P = (a, b)$. Set $\Delta \mathbf{v} = \langle h, k\rangle$. Show that the linear approximation can be written

$$\Delta f \approx \nabla f_P \cdot \Delta \mathbf{v} \qquad \boxed{7}$$

57. Use Eq. (7) to estimate

$$\Delta f = f(3.53, 8.98) - f(3.5, 9)$$

assuming that $\nabla f_{(3.5,9)} = \langle 2, -1\rangle$.

58. Find a unit vector $\mathbf{n}$ that is normal to the surface $z^2 - 2x^4 - y^4 = 16$ at $P = (2, 2, 8)$ that points in the direction of the xy-plane (in other words, if you travel in the direction of $\mathbf{n}$, you will eventually cross the xy-plane).

59. Suppose, in the previous exercise, that a particle located at the point $P = (2, 2, 8)$ travels toward the xy-plane in the direction normal to the surface.

(a) Through which point Q on the xy-plane will the particle pass?

(b) Suppose the axes are calibrated in centimeters. Determine the path $\mathbf{r}(t)$ of the particle if it travels at a constant speed of 8 cm/s. How long will it take the particle to reach Q?

60. Let $f(x, y) = \tan^{-1} \frac{x}{y}$ and $\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$.

(a) Calculate the gradient of f.

(b) Calculate $D_{\mathbf{u}} f(1, 1)$ and $D_{\mathbf{u}} f(\sqrt{3}, 1)$.

(c) Show that the lines $y = mx$ for $m \neq 0$ are level curves for f.

(d) Verify that ∇f_P is orthogonal to the level curve through P for $P = (x, y) \neq (0, 0)$.

61. Suppose that the intersection of two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ is a curve $\mathcal{C}$, and let P be a point on $\mathcal{C}$. Explain why the vector $\mathbf{v} = \nabla F_P \times \nabla G_P$ is a direction vector for the tangent line to $\mathcal{C}$ at P.

62. Let $\mathcal{C}$ be the curve of intersection of the spheres $x^2 + y^2 + z^2 = 3$ and $(x - 2)^2 + (y - 2)^2 + z^2 = 3$. Use the result of Exercise 61 to find parametric equations of the tangent line to $\mathcal{C}$ at $P = (1, 1, 1)$.

63. Let $\mathcal{C}$ be the curve obtained by intersecting the two surfaces $x^3 + 2xy + yz = 7$ and $3x^2 - yz = 1$. Find the parametric equations of the tangent line to $\mathcal{C}$ at $P = (1, 2, 1)$.

64. Verify the linearity relations for gradients:

(a) $\nabla(f + g) = \nabla f + \nabla g$

(b) $\nabla(cf) = c\nabla f$

65. Prove the Chain Rule for Gradients (Theorem 1).

66. Prove the Product Rule for Gradients (Theorem 1).

- Implicit differentiation is used to find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ when is defined implicitly by an equation $F(x, y, z) = 0$:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

14.6 EXERCISES

Preliminary Questions

1. Let $f(x, y) = xy$, where $x = uv$ and $y = u + v$.

(a) What are the primary derivatives of f?

(b) What are the independent variables?

In Questions 2 and 3, suppose that $f(u, v) = ue^v$, where $u = rs$ and $v = r + s$.

2. The composite function $f(u, v)$ is equal to:

(a) rse^{r+s} **(b)** re^s **(c)** rse^{rs}

3. What is the value of $f(u, v)$ at $(r, s) = (1, 1)$?

4. According to the Chain Rule, $\partial f/\partial r$ is equal to (choose the correct answer):

(a) $\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$

(b) $\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$

(c) $\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x}$

5. Suppose that x, y, z are functions of the independent variables u, v, w. Which of the following terms appear in the Chain Rule expression for $\partial f/\partial w$?

(a) $\frac{\partial f}{\partial v}\frac{\partial x}{\partial v}$ **(b)** $\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$ **(c)** $\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$

6. With notation as in the previous question, does $\partial x/\partial v$ appear in the Chain Rule expression for $\partial f/\partial u$?

Exercises

1. Let $f(x, y, z) = x^2y^3 + z^4$ and $x = s^2$, $y = st^2$, and $z = s^2t$.

(a) Calculate the primary derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

(b) Calculate $\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}$.

(c) Compute $\frac{\partial f}{\partial s}$ using the Chain Rule:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Express the answer in terms of the independent variables s, t.

2. Let $f(x, y) = x\cos(y)$ and $x = u^2 + v^2$ and $y = u - v$.

(a) Calculate the primary derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

(b) Use the Chain Rule to calculate $\partial f/\partial v$. Leave the answer in terms of both the dependent and the independent variables.

(c) Determine (x, y) for $(u, v) = (2, 1)$ and evaluate $\partial f/\partial v$ at $(u, v) = (2, 1)$.

In Exercises 3–10, use the Chain Rule to calculate the partial derivatives. Express the answer in terms of the independent variables.

3. $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}$; $f(x, y, z) = xy + z^2$, $x = s^2$, $y = 2rs$, $z = r^2$

4. $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}$; $f(x, y, z) = xy + z^2$, $x = r + s - 2t$, $y = 3rt$, $z = s^2$

5. $\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}$; $g(x, y) = \cos(x - y)$, $x = 3u - 5v$, $y = -7u + 15v$

6. $\frac{\partial R}{\partial u}, \frac{\partial R}{\partial v}$; $R(x, y) = (3x + 4y)^5$, $x = u^2$, $y = uv$

7. $\frac{\partial F}{\partial y}$; $F(u, v) = e^{u+v}$, $u = x^2$, $v = xy$

8. $\frac{\partial f}{\partial u}$; $f(x, y) = x^2 + y^2$, $x = e^{u+v}$, $y = u + v$

9. $\frac{\partial h}{\partial t_2}$; $h(x, y) = \frac{x}{y}$, $x = t_1t_2$, $y = t_1^2t_2$

10. $\frac{\partial f}{\partial \theta}$; $f(x, y, z) = xy - z^2$, $x = r\cos\theta$, $y = \cos^2\theta$, $z = r$

In Exercises 11–16, use the Chain Rule to evaluate the partial derivative at the point specified.

11. $\partial f/\partial u$ and $\partial f/\partial v$ at $(u, v) = (-1, -1)$, where $f(x, y, z) = x^3 + yz^2$, $x = u^2 + v$, $y = u + v^2$, $z = uv$

12. $\partial f/\partial s$ at $(r, s) = (1, 0)$, where $f(x, y) = \ln(xy)$, $x = 3r + 2s$, $y = 5r + 3s$

13. $\partial g/\partial \theta$ at $(r, \theta) = (2\sqrt{2}, \frac{\pi}{4})$, where $g(x, y) = 1/(x + y^2)$, $x = r\cos\theta$, $y = r\sin\theta$

14. dg/ds at $s = 4$, where $g(x, y) = x^2 - y^2$, $x = s^2 + 1$, $y = 1 - 2s$

15. $\partial g/\partial u$ at $(u, v) = (0, 1)$, where $g(x, y) = x^2 - y^2$, $x = e^u\cos v$, $y = e^u\sin v$

16. $\frac{\partial h}{\partial q}$ at $(q, r) = (3, 2)$, where $h(u, v) = ue^v$, $u = q^3$, $v = qr^2$

17. A baseball player hits the ball and then runs down the first base line at 20 ft/s. The first baseman fields the ball and then runs toward first base along the second base line at 18 ft/s as in Figure 7.

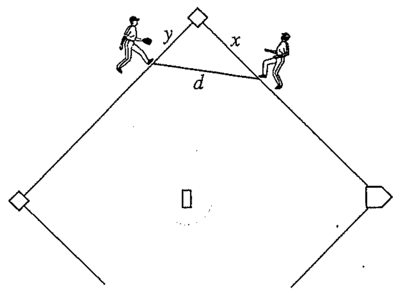


FIGURE 7

Determine how fast the distance between the two players is changing at a moment when the hitter is 8 ft from first base and the first baseman is 6 ft from first base.

18. Jessica and Matthew are running toward the point P along the straight paths that make a fixed angle of θ (Figure 8). Suppose that Matthew runs with velocity v_a meters per second and Jessica with velocity v_b meters per second. Let $f(x, y)$ be the distance from Matthew to Jessica when Matthew is x meters from P and Jessica is y meters from P.

(a) Show that $f(x, y) = \sqrt{x^2 + y^2 - 2xy\cos\theta}$.

(b) Assume that $\theta = \pi/3$. Use the Chain Rule to determine the rate at which the distance between Matthew and Jessica is changing when $x = 30$, $y = 20$, $v_a = 4$ m/s, and $v_b = 3$ m/s.

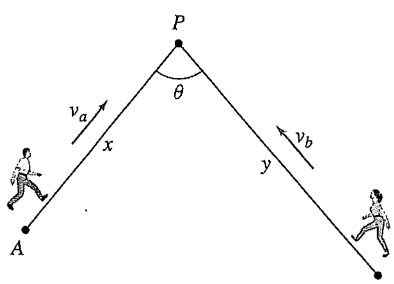


FIGURE 8

19. Two spacecraft are following paths in space given by $\mathbf{r}_1 = \langle \sin t, t, t^2 \rangle$ and $\mathbf{r}_2 = \langle \cos t, 1 - t, t^3 \rangle$ If the temperature for points in space are given by $T(x, y, z) = x^2 y(1 - z)$, use the Chain Rule to determine the rate of change of the difference D in the temperatures the two spacecraft experience at time $t = \pi$.

20. The Law of Cosines states that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where a, b, c are the sides of a triangle and θ is the angle opposite the side of length c.

(a) Compute $\partial\theta/\partial a$, $\partial\theta/\partial b$, and $\partial\theta/\partial c$ using implicit differentiation.

(b) Suppose that $a = 10$, $b = 16$, $c = 22$. Estimate the change in θ if a and b are increased by 1 and c is increased by 2.

21. Let $u = u(x, y)$, and let (r, θ) be polar coordinates. Verify the relation

$$\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2}u_\theta^2 \qquad \boxed{8}$$

Hint: Compute the right-hand side by expressing u_θ and u_r in terms of u_x and u_y.

22. Let $u(r, \theta) = r^2\cos^2\theta$. Use Eq. (8) to compute $\|\nabla u\|^2$. Then compute $\|\nabla u\|^2$ directly by observing that $u(x, y) = x^2$, and compare.

23. Let $x = s + t$ and $y = s - t$. Show that for any differentiable function $f(x, y)$,

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial s}\frac{\partial f}{\partial t}$$

24. Express the derivatives

$$\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \quad \text{in terms of} \quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

where (ρ, θ, ϕ) are spherical coordinates.

25. Suppose that z is defined implicitly as a function of x and y by the equation $F(x, y, z) = xz^2 + y^2 z + xy - 1 = 0$.

(a) Calculate F_x, F_y, F_z.

(b) Use Eq. (7) to calculate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

26. Calculate $\partial z/\partial x$ and $\partial z/\partial y$ at the points $(3, 2, 1)$ and $(3, 2, -1)$, where z is defined implicitly by the equation $z^4 + z^2 x^2 - y - 8 = 0$.

In Exercises 27–32, calculate the partial derivative using implicit differentiation.

27. $\dfrac{\partial z}{\partial x}$, $\quad x^2 y + y^2 z + xz^2 = 10$

28. $\dfrac{\partial w}{\partial z}$, $\quad x^2 w + w^3 + wz^2 + 3yz = 0$

29. $\dfrac{\partial z}{\partial y}$, $\quad e^{xy} + \sin(xz) + y = 0$

30. $\dfrac{\partial r}{\partial t}$ and $\dfrac{\partial t}{\partial r}$, $\quad r^2 = te^{s/r}$

31. $\dfrac{\partial w}{\partial y}$, $\quad \dfrac{1}{w^2 + x^2} + \dfrac{1}{w^2 + y^2} = 1$ at $(x, y, w) = (1, 1, 1)$

32. $\partial U/\partial T$ and $\partial T/\partial U$, $\quad (TU - V)^2 \ln(W - UV) = 1$ at $(T, U, V, W) = (1, 1, 2, 4)$

33. Let $\mathbf{r} = \langle x, y, z \rangle$ and $e_{\mathbf{r}} = \mathbf{r}/\|\mathbf{r}\|$. Show that if a function $f(x, y, z) = F(r)$ depends only on the distance from the origin $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$, then

$$\nabla f = F'(r)e_{\mathbf{r}} \qquad \boxed{9}$$

34. Let $f(x, y, z) = e^{-x^2-y^2-z^2} = e^{-r^2}$, with r as in Exercise 33. Compute ∇f directly and using Eq. (9).

35. Use Eq. (9) to compute $\nabla\left(\dfrac{1}{r}\right)$.

36. Use Eq. (9) to compute $\nabla(\ln r)$.

37. Figure 9 shows the graph of the equation

$$F(x, y, z) = x^2 + y^2 - z^2 - 12x - 8z - 4 = 0$$

(a) Use the quadratic formula to solve for z as a function of x and y. This gives two formulas, depending on the choice of sign.

(b) Which formula defines the portion of the surface satisfying $z \geq -4$? Which formula defines the portion satisfying $z \leq -4$?

14.7 EXERCISES

Preliminary Questions

1. The functions $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 - y^2$ both have a critical point at $(0, 0)$. How is the behavior of the two functions at the critical point different?

2. Identify the points indicated in the contour maps as local minima, local maxima, saddle points, or neither (Figure 17).

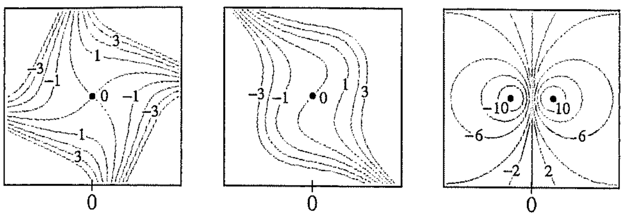


FIGURE 17

3. Let $f(x, y)$ be a continuous function on a domain $\mathcal{D}$ in $\mathbf{R}^2$. Determine which of the following statements are true:

(a) If $\mathcal{D}$ is closed and bounded, then f takes on a maximum value on $\mathcal{D}$.

(b) If $\mathcal{D}$ is neither closed nor bounded, then f does not take on a maximum value of $\mathcal{D}$.

(c) $f(x, y)$ need not have a maximum value on the domain $\mathcal{D}$ defined by $0 \le x \le 1, 0 \le y \le 1$.

(d) A continuous function takes on neither a minimum nor a maximum value on the open quadrant

$$\{(x, y) : x > 0, y > 0\}$$

Exercises

1. Let $P = (a, b)$ be a critical point of $f(x, y) = x^2 + y^4 - 4xy$.
(a) First use $f_x(x, y) = 0$ to show that $a = 2b$. Then use $f_y(x, y) = 0$ to show that $P = (0, 0)$, $(2\sqrt{2}, \sqrt{2})$, or $(-2\sqrt{2}, -\sqrt{2})$.
(b) Referring to Figure 18, determine the local minima and saddle points of $f(x, y)$ and find the absolute minimum value of $f(x, y)$.

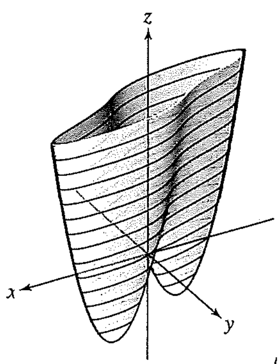
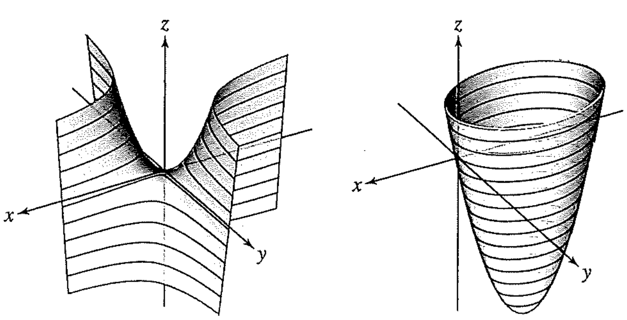


FIGURE 18

2. Find the critical points of the functions

$$f(x, y) = x^2 + 2y^2 - 4y + 6x, \qquad g(x, y) = x^2 - 12xy + y$$

Use the Second Derivative Test to determine the local minimum, local maximum, and saddle points. Match $f(x, y)$ and $g(x, y)$ with their graphs in Figure 19.



(A) (B)

FIGURE 19

3. Find the critical points of

$$f(x, y) = 8y^4 + x^2 + xy - 3y^2 - y^3$$

Use the contour map in Figure 20 to determine their nature (local minimum, local maximum, or saddle point).

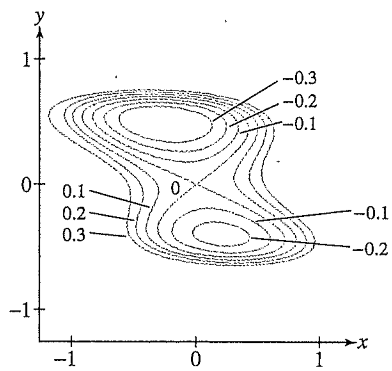


FIGURE 20 Contour map of $f(x, y) = 8y^4 + x^2 + xy - 3y^2 - y^3$.

4. Use the contour map in Figure 21 to determine whether the critical points A, B, C, D are local minima, local maxima, or saddle points.

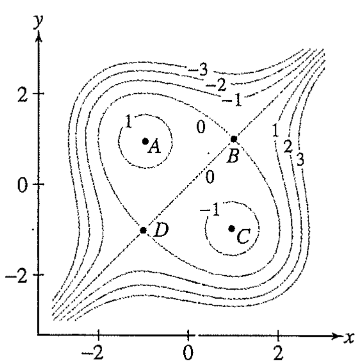


FIGURE 21

5. Let $f(x, y) = y^2x - yx^2 + xy$.
(a) Show that the critical points (x, y) satisfy the equations

$$y(y - 2x + 1) = 0, \qquad x(2y - x + 1) = 0$$

(b) Show that f has three critical points where $x = 0$ or $y = 0$ (or both) and one critical point where x and y are nonzero.

(c) Use the Second Derivative Test to determine the nature of the critical points.

6. Show that $f(x, y) = \sqrt{x^2 + y^2}$ has one critical point P and that f is nondifferentiable at P. Does f have a minimum, maximum, or saddle point at P?

In Exercises 7–23, find the critical points of the function. Then use the Second Derivative Test to determine whether they are local minima, local maxima, or saddle points (or state that the test fails).

7. $f(x, y) = x^2 + y^2 - xy + x$

8. $f(x, y) = x^3 - xy + y^3$

9. $f(x, y) = x^3 + 2xy - 2y^2 - 10x$

10. $f(x, y) = x^3y + 12x^2 - 8y$

11. $f(x, y) = 4x - 3x^3 - 2xy^2$

12. $f(x, y) = x^3 + y^4 - 6x - 2y^2$

13. $f(x, y) = x^4 + y^4 - 4xy$

14. $f(x, y) = e^{x^2 - y^2 + 4y}$

15. $f(x, y) = xye^{-x^2 - y^2}$

16. $f(x, y) = e^x - xe^y$

17. $f(x, y) = \sin(x + y) - \cos x$

18. $f(x, y) = x \ln(x + y)$

19. $f(x, y) = \ln x + 2 \ln y - x - 4y$

20. $f(x, y) = (x + y) \ln(x^2 + y^2)$

21. $f(x, y) = x - y^2 - \ln(x + y)$

22. $f(x, y) = (x - y)e^{x^2 - y^2}$

23. $f(x, y) = (x + 3y)e^{y - x^2}$

24. Show that $f(x, y) = x^2$ has infinitely many critical points (as a function of two variables) and that the Second Derivative Test fails for all of them. What is the minimum value of f? Does $f(x, y)$ have any local maxima?

25. Prove that the function $f(x, y) = \frac{1}{3}x^3 + \frac{2}{3}y^{3/2} - xy$ satisfies $f(x, y) \geq 0$ for $x \geq 0$ and $y \geq 0$.

(a) First, verify that the set of critical points of f is the parabola $y = x^2$ and that the Second Derivative Test fails for these points.

(b) Show that for fixed b, the function $g(x) = f(x, b)$ is concave up for $x > 0$ with a critical point at $x = b^{1/2}$.

(c) Conclude that $f(a, b) \geq f(b^{1/2}, b) = 0$ for all $a, b \geq 0$.

26. Let $f(x, y) = (x^2 + y^2)e^{-x^2 - y^2}$.

(a) Where does f take on its minimum value? Do not use calculus to answer this question.

(b) Verify that the set of critical points of f consists of the origin $(0, 0)$ and the unit circle $x^2 + y^2 = 1$.

(c) The Second Derivative Test fails for points on the unit circle (this can be checked by some lengthy algebra). Prove, however, that f takes on its maximum value on the unit circle by analyzing the function $g(t) = te^{-t}$ for $t > 0$.

27. CAS Use a computer algebra system to find a numerical approximation to the critical point of

$$f(x, y) = (1 - x + x^2)e^{y^2} + (1 - y + y^2)e^{x^2}$$

Apply the Second Derivative Test to confirm that it corresponds to a local minimum as in Figure 22.

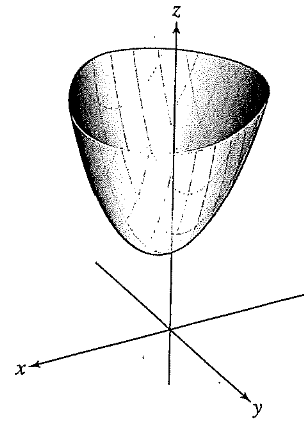


FIGURE 22 Plot of $f(x, y) = (1 - x + x^2)e^{y^2} + (1 - y + y^2)e^{x^2}$.

28. Which of the following domains are closed and which are bounded?

(a) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$

(b) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$

(c) $\{(x, y) \in \mathbf{R}^2 : x \geq 0\}$

(d) $\{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}$

(e) $\{(x, y) \in \mathbf{R}^2 : 1 \leq x \leq 4, 5 \leq y \leq 10\}$

(f) $\{(x, y) \in \mathbf{R}^2 : x > 0, x^2 + y^2 \leq 10\}$

In Exercises 29–32, determine the global extreme values of the function on the given set without using calculus.

29. $f(x, y) = x + y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$

30. $f(x, y) = 2x - y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 3$

31. $f(x, y) = (x^2 + y^2 + 1)^{-1}, \quad 0 \leq x \leq 3, \quad 0 \leq y \leq 5$

32. $f(x, y) = e^{-x^2 - y^2}, \quad x^2 + y^2 \leq 1$

33. Assumptions Matter Show that $f(x, y) = xy$ does not have a global minimum or a global maximum on the domain

$$\mathcal{D} = \{(x, y) : 0 < x < 1, 0 < y < 1\}$$

Explain why this does not contradict Theorem 3.

34. Find a continuous function that does not have a global maximum on the domain $\mathcal{D} = \{(x, y) : x + y \geq 0, x + y \leq 1\}$. Explain why this does not contradict Theorem 3.

35. Find the maximum of

$$f(x, y) = x + y - x^2 - y^2 - xy$$

on the square, $0 \leq x \leq 2, 0 \leq y \leq 2$ (Figure 23).

(a) First, locate the critical point of f in the square, and evaluate f at this point.

(b) On the bottom edge of the square, $y = 0$ and $f(x, 0) = x - x^2$. Find the extreme values of f on the bottom edge.

(c) Find the extreme values of f on the remaining edges.

(d) Find the largest among the values computed in (a), (b), and (c).

Exercises

In this exercise set, use the method of Lagrange multipliers unless otherwise stated.

1. Find extreme values of the function $f(x, y) = 2x + 4y$ subject to the constraint $g(x, y) = x^2 + y^2 - 5 = 0$.

(a) Show that the Lagrange equation $\nabla f = \lambda \nabla g$ gives $\lambda x = 1$ and $\lambda y = 2$.

(b) Show that these equations imply $\lambda \neq 0$ and $y = 2x$.

(c) Use the constraint equation to determine the possible critical points (x, y).

(d) Evaluate $f(x, y)$ at the critical points and determine the minimum and maximum values.

2. Find the extreme values of $f(x, y) = x^2 + 2y^2$ subject to the constraint $g(x, y) = 4x - 6y = 25$.

(a) Show that the Lagrange equations yield $2x = 4\lambda$, $4y = -6\lambda$.

(b) Show that if $x = 0$ or $y = 0$, then the Lagrange equations give $x = y = 0$. Since $(0, 0)$ does not satisfy the constraint, you may assume that x and y are nonzero.

(c) Use the Lagrange equations to show that $y = -\frac{3}{4}x$.

(d) Substitute in the constraint equation to show that there is a unique critical point P.

(e) Does P correspond to a minimum or maximum value of f? Refer to Figure 13 to justify your answer. *Hint:* Do the values of $f(x, y)$ increase or decrease as (x, y) moves away from P along the line $g(x, y) = 0$?

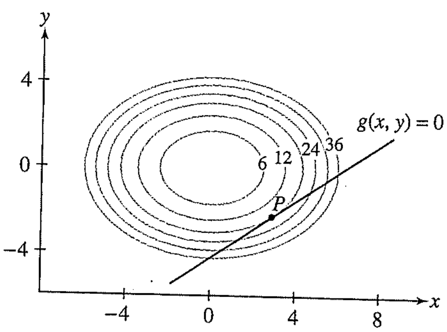


FIGURE 13 Level curves of $f(x, y) = x^2 + 2y^2$ and graph of the constraint $g(x, y) = 4x - 6y - 25 = 0$.

3. Apply the method of Lagrange multipliers to the function $f(x, y) = (x^2 + 1)y$ subject to the constraint $x^2 + y^2 = 5$. *Hint:* First show that $y \neq 0$; then treat the cases $x = 0$ and $x \neq 0$ separately.

In Exercises 4–15, find the minimum and maximum values of the function subject to the given constraint.

4. $f(x, y) = 2x + 3y, \quad x^2 + y^2 = 4$

5. $f(x, y) = x^2 + y^2, \quad 2x + 3y = 6$

6. $f(x, y) = 4x^2 + 9y^2, \quad xy = 4$

7. $f(x, y) = xy, \quad 4x^2 + 9y^2 = 32$

8. $f(x, y) = x^2y + x + y, \quad xy = 4$

9. $f(x, y) = x^2 + y^2, \quad x^4 + y^4 = 1$

10. $f(x, y) = x^2y^4, \quad x^2 + 2y^2 = 6$

11. $f(x, y, z) = 3x + 2y + 4z, \quad x^2 + 2y^2 + 6z^2 = 1$

12. $f(x, y, z) = x^2 - y - z, \quad x^2 - y^2 + z = 0$

13. $f(x, y, z) = xy + 2z, \quad x^2 + y^2 + z^2 = 36$

14. $f(x, y, z) = x^2 + y^2 + z^2, \quad x + 3y + 2z = 36$

15. $f(x, y, z) = xy + xz, \quad x^2 + y^2 + z^2 = 4$

16. Let

$$f(x, y) = x^3 + xy + y^3, \qquad g(x, y) = x^3 - xy + y^3$$

(a) Show that there is a unique point $P = (a, b)$ on $g(x, y) = 1$ where $\nabla f_P = \lambda \nabla g_P$ for some scalar λ.

(b) Refer to Figure 14 to determine whether $f(P)$ is a local minimum or a local maximum of f subject to the constraint.

(c) Does Figure 14 suggest that $f(P)$ is a global extremum subject to the constraint?

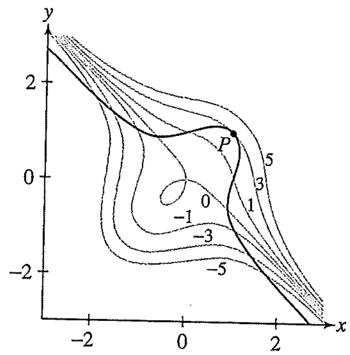


FIGURE 14 Contour map of $f(x, y) = x^3 + xy + y^3$ and graph of the constraint $g(x, y) = x^3 - xy + y^3 = 1$.

17. Find the point (a, b) on the graph of $y = e^x$ where the value ab is as small as possible.

18. Find the rectangular box of maximum volume if the sum of the lengths of the edges is 300 cm.

19. The surface area of a right-circular cone of radius r and height h is $S = \pi r\sqrt{r^2 + h^2}$, and its volume is $V = \frac{1}{3}\pi r^2 h$.

(a) Determine the ratio h/r for the cone with given surface area S and maximum volume V.

(b) What is the ratio h/r for a cone with given volume V and minimum surface area S?

(c) Does a cone with given volume V and maximum surface area exist?

20. In Example 1, we found the maximum of $f(x, y) = 2x + 5y$ on the ellipse $(x/4)^2 + (y/3)^2 = 1$. Solve this problem again without using Lagrange multipliers. First, show that the ellipse is parametrized by $x = 4\cos t$, $y = 3\sin t$. Then find the maximum value of $f(4\cos t, 3\sin t)$ using single-variable calculus. Is one method easier than the other?

21. Find the point on the ellipse

$$x^2 + 6y^2 + 3xy = 40$$

with the largest x-coordinate (Figure 15).

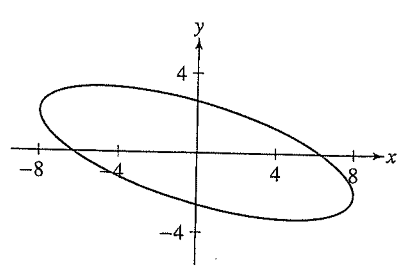


FIGURE 15 Graph of $x^2 + 6y^2 + 3xy = 40$.

22. Use Lagrange multipliers to find the maximum area of a rectangle inscribed in the ellipse (Figure 16):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

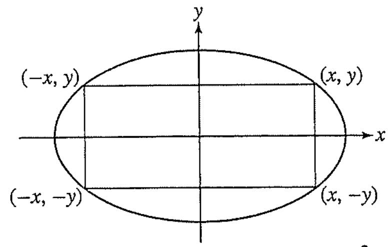


FIGURE 16 Rectangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

23. Find the point (x_0, y_0) on the line $4x + 9y = 12$ that is closest to the origin.

24. Show that the point (x_0, y_0) closest to the origin on the line $ax + by = c$ has coordinates

$$x_0 = \frac{ac}{a^2 + b^2}, \qquad y_0 = \frac{bc}{a^2 + b^2}$$

25. Find the maximum value of $f(x, y) = x^a y^b$ for $x \geq 0, y \geq 0$ on the line $x + y = 1$, where $a, b > 0$ are constants.

26. Show that the maximum value of $f(x, y) = x^2 y^3$ on the unit circle is $\frac{6}{25}\sqrt{\frac{3}{5}}$.

27. Find the maximum value of $f(x, y) = x^a y^b$ for $x \geq 0, y \geq 0$ on the unit circle, where $a, b > 0$ are constants.

28. Find the maximum value of $f(x, y, z) = x^a y^b z^c$ for $x, y, z \geq 0$ on the unit sphere, where $a, b, c > 0$ are constants.

29. Show that the minimum distance from the origin to a point on the plane $ax + by + cz = d$ is

$$\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

30. Antonio has \$5.00 to spend on a lunch consisting of hamburgers (\$1.50 each) and french fries (\$1.00 per order). Antonio's satisfaction from eating x_1 hamburgers and x_2 orders of french fries is measured by a function $U(x_1, x_2) = \sqrt{x_1 x_2}$. How much of each type of food should he purchase to maximize his satisfaction? (Assume that fractional amounts of each food can be purchased.)

31. Let Q be the point on an ellipse closest to a given point P outside the ellipse. It was known to the Greek mathematician Apollonius (third century BCE) that $\overline{PQ}$ is perpendicular to the tangent to the ellipse at Q (Figure 17). Explain in words why this conclusion is a consequence of the method of Lagrange multipliers. *Hint:* The circles centered at P are level curves of the function to be minimized.

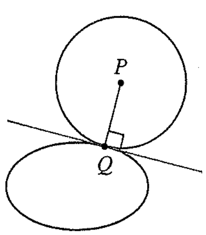


FIGURE 17

32. In a contest, a runner starting at A must touch a point P along a river and then run to B in the shortest time possible (Figure 18). The runner should choose the point P that minimizes the total length of the path.

(a) Define a function

$$f(x, y) = AP + PB, \quad \text{where } P = (x, y)$$

Rephrase the runner's problem as a constrained optimization problem, assuming that the river is given by an equation $g(x, y) = 0$.

(b) Explain why the level curves of $f(x, y)$ are ellipses.

(c) Use Lagrange multipliers to justify the following statement: The ellipse through the point P minimizing the length of the path is tangent to the river.

(d) Identify the point on the river in Figure 18 for which the length is minimal.

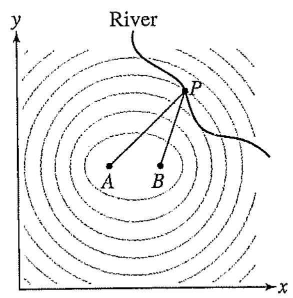


FIGURE 18

In Exercises 33 and 34, let V be the volume of a can of radius r and height h, and let S be its surface area (including the top and bottom).

33. Find r and h that minimize S subject to the constraint $V = 54\pi$.

34. Show that for both of the following two problems, $P = (r, h)$ is a Lagrange critical point if $h = 2r$:

- Minimize surface area S for fixed volume V.
- Maximize volume V for fixed surface area S.

Then use the contour plots in Figure 19 to explain why S has a minimum for fixed V but no maximum and, similarly, V has a maximum for fixed S but no minimum.

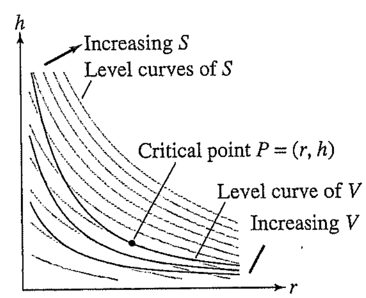


FIGURE 19

35. A plane with equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $(a, b, c > 0)$ together w the positive coordinate planes forms a tetrahedron of volume $V = \frac{1}{6}$ (Figure 20). Find the minimum value of V among all planes pa through the point $P = (1, 1, 1)$.