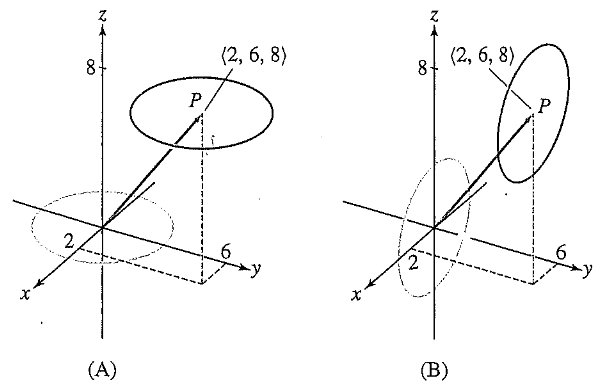

FIGURE 7 Horizontal and vertical circles of radius 3 and center $P = (2, 6, 8)$ obtained by translation.

13.1 SUMMARY

- A *vector-valued function* is a function of the form

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

- We often think of t as time and $\mathbf{r}(t)$ as a "moving vector" whose terminal point traces out a path as a function of time. We refer to $\mathbf{r}(t)$ as a *vector parametrization* of the path, or simply as a "path."
- The underlying curve C traced by $\mathbf{r}(t)$ is the set of all points $(x(t), y(t), z(t))$ in $\mathbf{R}^3$ for t in the domain of $\mathbf{r}(t)$. A curve in $\mathbf{R}^3$ is also called a *space curve*.
- Every curve C can be parametrized in infinitely many ways.
- The projection of $\mathbf{r}(t)$ onto the xy-plane is the curve traced by $\langle x(t), y(t), 0 \rangle$. The projection onto the xz-plane is $\langle x(t), 0, z(t) \rangle$, and the projection onto the yz-plane is $\langle 0, y(t), z(t) \rangle$.

13.1 EXERCISES

Preliminary Questions

1. Which one of the following does *not* parametrize a line?
(a) $\mathbf{r}_1(t) = \langle 8 - t, 2t, 3t \rangle$
(b) $\mathbf{r}_2(t) = t^3\mathbf{i} - 7t^3\mathbf{j} + t^3\mathbf{k}$
(c) $\mathbf{r}_3(t) = \langle 8 - 4t^3, 2 + 5t^2, 9t^3 \rangle$

2. What is the projection of $\mathbf{r}(t) = t\mathbf{i} + t^4\mathbf{j} + e^t\mathbf{k}$ onto the xz-plane?

3. Which projection of $\langle \cos t, \cos 2t, \sin t \rangle$ is a circle?

4. What is the center of the circle with the following parametrization?

$$\mathbf{r}(t) = (-2 + \cos t)\mathbf{i} + 2\mathbf{j} + (3 - \sin t)\mathbf{k}$$

5. How do the paths $\mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \cos t \rangle$ around the unit circle differ?

6. Which three of the following vector-valued functions parametrize the same space curve?
(a) $(-2 + \cos t)\mathbf{i} + 9\mathbf{j} + (3 - \sin t)\mathbf{k}$
(b) $(2 + \cos t)\mathbf{i} - 9\mathbf{j} + (-3 - \sin t)\mathbf{k}$
(c) $(-2 + \cos 3t)\mathbf{i} + 9\mathbf{j} + (3 - \sin 3t)\mathbf{k}$
(d) $(-2 - \cos t)\mathbf{i} + 9\mathbf{j} + (3 + \sin t)\mathbf{k}$
(e) $(2 + \cos t)\mathbf{i} + 9\mathbf{j} + (3 + \sin t)\mathbf{k}$

Exercises

1. What is the domain of $\mathbf{r}(t) = e^t\mathbf{i} + \frac{1}{t}\mathbf{j} + (t+1)^{-3}\mathbf{k}$?

2. What is the domain of $\mathbf{r}(s) = e^s\mathbf{i} + \sqrt{s}\mathbf{j} + \cos s\mathbf{k}$?

3. Evaluate $\mathbf{r}(2)$ and $\mathbf{r}(-1)$ for $\mathbf{r}(t) = \left\langle \sin \frac{\pi}{2}t, t^2, (t^2+1)^{-1} \right\rangle$.

4. Does either of $P = (4, 11, 20)$ or $Q = (-1, 6, 16)$ lie on the path $\mathbf{r}(t) = \langle 1 + t, 2 + t^2, t^4 \rangle$?

5. Find a vector parametrization of the line through $P = (3, -5, 7)$ the direction $\mathbf{v} = \langle 3, 0, 1 \rangle$.

6. Find a direction vector for the line with parametrization $\mathbf{r}(t) = (4 - t)\mathbf{i} + (2 + 5t)\mathbf{j} + \frac{1}{2}t\mathbf{k}$.

7. Determine whether the space curve given by $\mathbf{r}(t) = \langle \sin t, \cos t/2, t \rangle$ intersects the z-axis, and if it does, determine where.

8. Determine whether the space curve given by $\mathbf{r}(t) = \langle t^2, t^2 - 2t - 3, t - 3 \rangle$ intersects the x-axis, and if it does, determine where.

9. Determine whether the space curve given by $\mathbf{r}(t) = \left\langle t, t^3, t^2 + 1\right\rangle$ intersects the xy-plane, and if it does, determine where.

10. Determine whether the space curves given by $\mathbf{r}_1(t) = \left\langle t, t^2, t + 1\right\rangle$ and $\mathbf{r}_2(s) = \left\langle \sqrt{s}, s, s - 1\right\rangle$ intersect, and if they do, determine where.

11. Match the space curves in Figure 8 with their projections onto the xy-plane in Figure 9.

12. Match the space curves in Figure 8 with the following vector-valued functions:

(a) $\mathbf{r}_1(t) = \langle \cos 2t, \cos t, \sin t\rangle$ **(b)** $\mathbf{r}_2(t) = \langle t, \cos 2t, \sin 2t\rangle$

(c) $\mathbf{r}_3(t) = \langle 1, t, t\rangle$

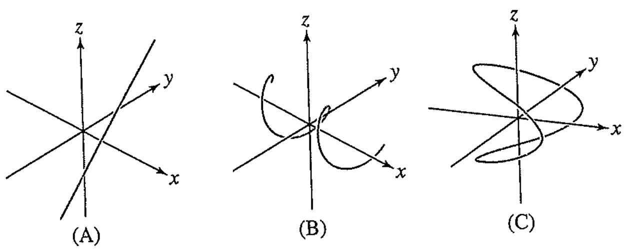

FIGURE 8

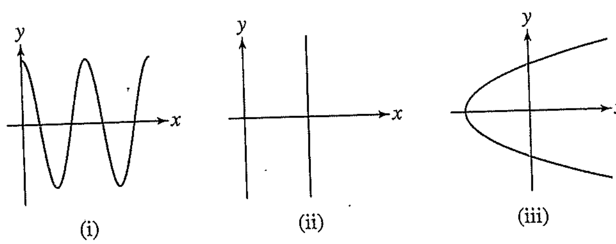

FIGURE 9

13. Match the vector-valued functions (a)–(f) with the space curves (i)–(vi) in Figure 10.

(a) $\mathbf{r}(t) = \left\langle t + 15, e^{0.08t}\cos t, e^{0.08t}\sin t\right\rangle$

(b) $\mathbf{r}(t) = \left\langle \cos t, \sin t, \sin 12t\right\rangle$ **(c)** $\mathbf{r}(t) = \left\langle t, t, \dfrac{25t}{1 + t^2}\right\rangle$

(d) $\mathbf{r}(t) = \left\langle \cos^3 t, \sin^3 t, \sin 2t\right\rangle$ **(e)** $\mathbf{r}(t) = \left\langle t, t^2, 2t\right\rangle$

(f) $\mathbf{r}(t) = \left\langle \cos t, \sin t, \cos t \sin 12t\right\rangle$

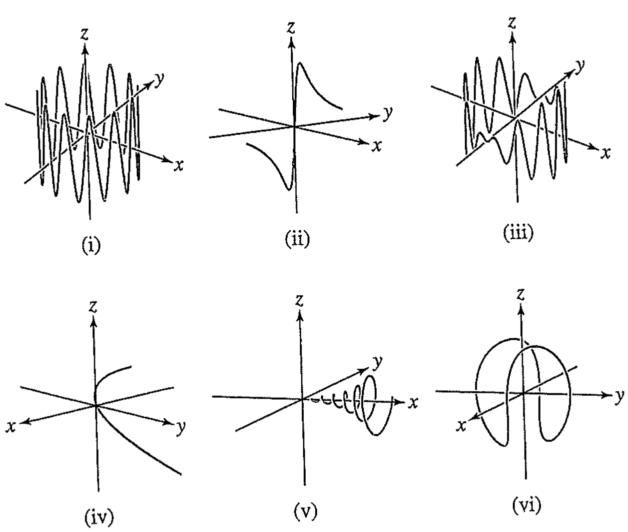

FIGURE 10

14. Which of the following curves have the same projection onto the xy-plane?

(a) $\mathbf{r}_1(t) = \left\langle t, t^2, e^t\right\rangle$ **(b)** $\mathbf{r}_2(t) = \left\langle e^t, t^2, t\right\rangle$

(c) $\mathbf{r}_3(t) = \left\langle t, t^2, \cos t\right\rangle$

15. Match the space curves (A)–(C) in Figure 11 with their projections (i)–(iii) onto the xy-plane.

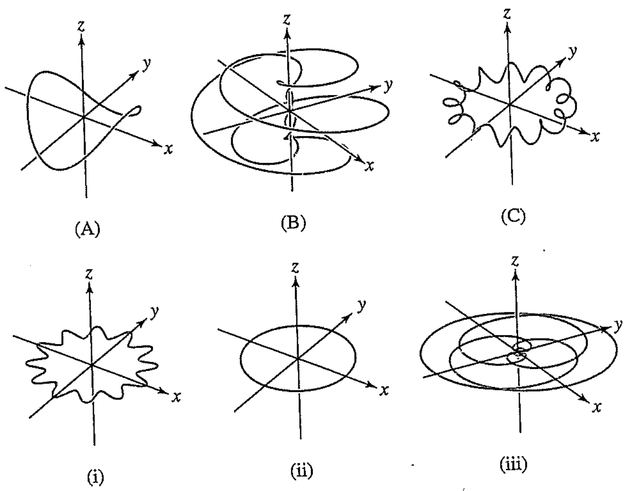

FIGURE 11

16. Describe the projections of the circle $\mathbf{r}(t) = \langle \sin t, 0, 4 + \cos t\rangle$ onto the coordinate planes.

In Exercises 17–20, the function $\mathbf{r}(t)$ *traces a circle. Determine the radius, center, and plane containing the circle.*

17. $\mathbf{r}(t) = (9\cos t)\mathbf{i} + (9\sin t)\mathbf{j}$

18. $\mathbf{r}(t) = 7\mathbf{i} + (12\cos t)\mathbf{j} + (12\sin t)\mathbf{k}$

19. $\mathbf{r}(t) = \langle \sin t, 0, 4 + \cos t\rangle$

20. $\mathbf{r}(t) = \langle 6 + 3\sin t, 9, 4 + 3\cos t\rangle$

21. Let C be the curve given by $\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$.

(a) Show that C lies on the cone $x^2 + y^2 = z^2$.

(b) Sketch the cone and make a rough sketch of C on the cone.

22. CAS Use a computer algebra system to plot the projections onto the xy- and xz-planes of the curve $\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$ in Exercise 21.

In Exercises 23 and 24, let

$$\mathbf{r}(t) = \langle \sin t, \cos t, \sin t \cos 2t\rangle$$

be the parametrization of the curve shown in Figure 12.

23. Find the points where $\mathbf{r}(t)$ intersects the xy-plane.

24. Show that the projection of $\mathbf{r}(t)$ onto the xz-plane is the curve

$$z = x - 2x^3 \quad \text{for} \quad -1 \le x \le 1$$

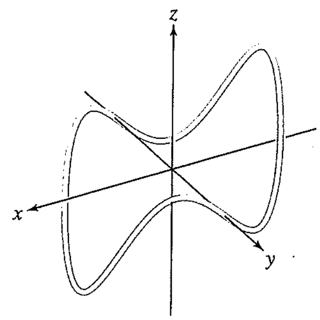

FIGURE 12

25. Parametrize the intersection of the surfaces

$$y^2 - z^2 = x - 2, \qquad y^2 + z^2 = 9$$

using $t = y$ as the parameter (two vector functions are needed as in Example 3).

26. Find a parametrization of the curve in Exercise 25 using trigonometric functions.

27. **Viviani's Curve** C is the intersection of the surfaces (Figure 13)

$$x^2 + y^2 = z^2, \qquad y = z^2$$

(a) Parametrize each of the two parts of C corresponding to $x \geq 0$ and $x \leq 0$, taking $t = z$ as the parameter.
(b) Describe the projection of C onto the xy-plane.
(c) Show that C lies on the sphere of radius 1 with its center $(0, 1, 0)$. This curve looks like a figure eight lying on a sphere [Figure 13(B)].

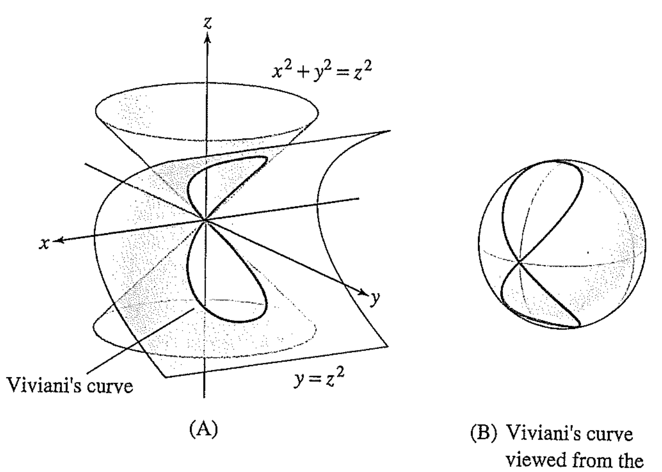

FIGURE 13 Viviani's curve is the intersection of the surfaces $x^2 + y^2 = z^2$ and $y = z^2$.

28. (a) Show that any point on $x^2 + y^2 = z^2$ can be written in the form $(z\cos\theta, z\sin\theta, z)$ for some θ.
(b) Use this to find a parametrization of Viviani's curve (Exercise 27) with θ as the parameter.

9. Use sine and cosine to parametrize the intersection of the cylinders $^2 + y^2 = 1$ and $x^2 + z^2 = 1$ (use two vector-valued functions). Then ·scribe the projections of this curve onto the three coordinate planes.

30. Use hyperbolic functions to parametrize the intersection of the surfaces $x^2 - y^2 = 4$ and $z = xy$.

31. Use sine and cosine to parametrize the intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$ (Figure 14).

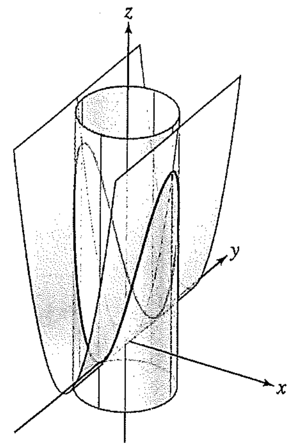

FIGURE 14 Intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$.

In Exercises 32–34, two paths $\mathbf{r}_1(t)$ *and* $\mathbf{r}_2(t)$ *intersect if there is a point P lying on both curves. We say that* $\mathbf{r}_1(t)$ *and* $\mathbf{r}_2(t)$ collide *if* $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$ *at some time* t_0.

32. Which of the following statements are true?
(a) If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect, then they collide.
(b) If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ collide, then they intersect.
(c) Intersection depends only on the underlying curves traced by $\mathbf{r}_1$ and $\mathbf{r}_2$, but collision depends on the actual parametrizations.

33. Determine whether $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ collide or intersect, giving the coordinates of the corresponding points if they exist:

$$\mathbf{r}_1(t) = \langle t^2 + 3, t + 1, 6t^{-1} \rangle$$
$$\mathbf{r}_2(t) = \langle 4t, 2t - 2, t^2 - 7 \rangle$$

34. Determine whether $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ collide or intersect, giving the coordinates of the corresponding points if they exist:

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle, \qquad \mathbf{r}_2(t) = \langle 4t + 6, 4t^2, 7 - t \rangle$$

In Exercises 35–44, find a parametrization of the curve.

35. The vertical line passing through the point $(3, 2, 0)$

36. The line passing through $(1, 0, 4)$ and $(4, 1, 2)$

37. The line through the origin whose projection on the xy-plane is a line of slope 3 and whose projection on the yz-plane is a line of slope 5 (i.e., $\Delta z/\Delta y = 5$)

38. The horizontal circle of radius 1 with center $(2, -1, 4)$

39. The circle of radius 2 with center $(1, 2, 5)$ in a plane parallel to the yz-plane

40. The ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ in the xy-plane, translated to have center $(9, -4, 0)$

41. The intersection of the plane $y = \frac{1}{2}$ with the sphere $x^2 + y^2 + z^2 = 1$

Exercises

In Exercises 1–6, evaluate the limit.

1. $\lim_{t\to 3}\left\langle t^2, 4t, \frac{1}{t}\right\rangle$

2. $\lim_{t\to\pi} \sin 2t\mathbf{i} + \cos t\mathbf{j} + \tan 4t\mathbf{k}$

3. $\lim_{t\to 0} e^{2t}\mathbf{i} + \ln(t+1)\mathbf{j} + 4\mathbf{k}$

4. $\lim_{t\to 0}\left\langle \frac{1}{t+1}, \frac{e^t-1}{t}, 4t\right\rangle$

5. Evaluate $\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ for $\mathbf{r}(t) = \left\langle t^{-1}, \sin t, 4\right\rangle$.

6. Evaluate $\lim_{t\to 0}\frac{\mathbf{r}(t)}{t}$ for $\mathbf{r}(t) = \langle \sin t, 1-\cos t, -2t\rangle$.

In Exercises 7–12, compute the derivative.

7. $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle$

8. $\mathbf{r}(t) = \left\langle 7-t, 4\sqrt{t}, 8\right\rangle$

9. $\mathbf{r}(s) = \left\langle e^{3s}, e^{-s}, s^4\right\rangle$

10. $\mathbf{b}(t) = \left\langle e^{3t-4}, e^{6-t}, (t+1)^{-1}\right\rangle$

11. $\mathbf{c}(t) = t^{-1}\mathbf{i} - e^{2t}\mathbf{k}$

12. $\mathbf{a}(\theta) = (\cos 3\theta)\mathbf{i} + (\sin^2\theta)\mathbf{j} + (\tan\theta)\mathbf{k}$

13. Calculate $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle$.

14. Sketch the curve parametrized by $\mathbf{r}(t) = \left\langle 1-t^2, t\right\rangle$ for $-1 \le t \le 1$. Compute the tangent vector at $t = 1$ and add it to the sketch.

15. Sketch the curve parametrized by $\mathbf{r}_1(t) = \left\langle t, t^2\right\rangle$ together with its tangent vector at $t = 1$. Then do the same for $\mathbf{r}_2(t) = \left\langle t^3, t^6\right\rangle$.

16. Sketch the cycloid $\mathbf{r}(t) = \left\langle t - \sin t, 1 - \cos t\right\rangle$ together with its tangent vectors at $t = \frac{\pi}{3}$ and $\frac{3\pi}{4}$.

In Exercises 17–20, evaluate the derivative by using the appropriate Product Rule, where

$$\mathbf{r}_1(t) = \left\langle t^2, t^3, t\right\rangle, \qquad \mathbf{r}_2(t) = \left\langle e^{3t}, e^{2t}, e^t\right\rangle$$

17. $\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t))$

18. $\frac{d}{dt}(t^4\mathbf{r}_1(t))$

19. $\frac{d}{dt}(\mathbf{r}_1(t)\times\mathbf{r}_2(t))$

20. $\frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{r}_1(t))\Big|_{t=2}$, assuming that

$$\mathbf{r}(2) = \langle 2, 1, 0\rangle, \qquad \mathbf{r}'(2) = \langle 1, 4, 3\rangle$$

In Exercises 21 and 22, let

$$\mathbf{r}_1(t) = \left\langle t^2, 1, 2t\right\rangle, \qquad \mathbf{r}_2(t) = \left\langle 1, 2, e^t\right\rangle$$

21. Compute $\frac{d}{dt}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\Big|_{t=1}$ in two ways:

(a) Calculate $\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)$ and differentiate.

(b) Use the Product Rule.

22. Compute $\frac{d}{dt}\mathbf{r}_1(t)\times\mathbf{r}_2(t)\Big|_{t=1}$ in two ways:

(a) Calculate $\mathbf{r}_1(t)\times\mathbf{r}_2(t)$ and differentiate.

(b) Use the Product Rule.

In Exercises 23–26, evaluate $\frac{d}{dt}\mathbf{r}(g(t))$ using the Chain Rule.

23. $\mathbf{r}(t) = \left\langle t^2, 1-t\right\rangle, \quad g(t) = e^t$

24. $\mathbf{r}(t) = \left\langle t^2, t^3\right\rangle, \quad g(t) = \sin t$

25. $\mathbf{r}(t) = \left\langle e^t, e^{2t}, 4\right\rangle, \quad g(t) = 4t + 9$

26. $\mathbf{r}(t) = \langle 4\sin 2t, 6\cos 2t\rangle, \quad g(t) = t^2$

27. Let $\mathbf{r}(t) = \left\langle t^2, 1-t, 4t\right\rangle$. Calculate the derivative of $\mathbf{r}(t)\cdot\mathbf{a}(t)$ at $t = 2$, assuming that $\mathbf{a}(2) = \langle 1, 3, 3\rangle$ and $\mathbf{a}'(2) = \langle -1, 4, 1\rangle$.

28. Let $\mathbf{v}(s) = s^2\mathbf{i} + 2s\mathbf{j} + 9s^{-2}\mathbf{k}$. Evaluate $\frac{d}{ds}\mathbf{v}(g(s))$ at $s = 4$, assuming that $g(4) = 3$ and $g'(4) = -9$.

In Exercises 29–34, find a parametrization of the tangent line at the point indicated.

29. $\mathbf{r}(t) = \left\langle t^2, t^4\right\rangle, \quad t = -2$

30. $\mathbf{r}(t) = \left\langle \cos 2t, \sin 3t\right\rangle, \quad t = \frac{\pi}{4}$

31. $\mathbf{r}(t) = \left\langle 1-t^2, 5t, 2t^3\right\rangle, \quad t = 2$

32. $\mathbf{r}(t) = \left\langle 4t, 5t, 9t\right\rangle, \quad t = -4$

33. $\mathbf{r}(s) = 4s^{-1}\mathbf{i} - \frac{8}{3}s^{-3}\mathbf{k}, \quad s = 2$

34. $\mathbf{r}(s) = (\ln s)\mathbf{i} + s^{-1}\mathbf{j} + 9s\mathbf{k}, \quad s = 1$

35. Use Example 4 to calculate $\frac{d}{dt}(\mathbf{r}\times\mathbf{r}')$, where $\mathbf{r}(t) = \left\langle t, t^2, e^t\right\rangle$.

36. Let $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t\rangle$. Show that $\|\mathbf{r}(t)\|$ is constant and conclude, using Example 7, that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Then compute $\mathbf{r}'(t)$ and verify directly that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

37. Show that the *derivative of the norm* is not equal to the *norm of the derivative* by verifying that $\|\mathbf{r}(t)\|' \neq \|\mathbf{r}(t)'\|$ for $\mathbf{r}(t) = \langle t, 1, 1\rangle$.

38. Show that $\frac{d}{dt}(\mathbf{a}\times\mathbf{r}) = \mathbf{a}\times\mathbf{r}'$ for any constant vector $\mathbf{a}$.

In Exercises 39–46, evaluate the integrals.

39. $\int_{-1}^{3}\left\langle 8t^2 - t, 6t^3 + t\right\rangle dt$

40. $\int_0^1\left\langle \frac{1}{1+s^2}, \frac{s}{1+s^2}\right\rangle ds$

41. $\int_{-2}^{2}\left(u^3\mathbf{i} + u^5\mathbf{j}\right) du$

42. $\int_0^1\left(te^{-t^2}\mathbf{i} + t\ln(t^2+1)\mathbf{j}\right) dt$

43. $\int_0^1 \langle 2t, 4t, -\cos 3t\rangle\, dt$

44. $\int_{1/2}^{1}\left\langle \frac{1}{u^2}, \frac{1}{u^4}, \frac{1}{u^5}\right\rangle du$

45. $\int_1^4\left(t^{-1}\mathbf{i} + 4\sqrt{t}\,\mathbf{j} - 8t^{3/2}\mathbf{k}\right) dt$

46. $\int_0^t\left(3s\mathbf{i} + 6s^2\mathbf{j} + 9\mathbf{k}\right) ds$

In Exercises 47–54, find both the general solution of the differential equation and the solution with the given initial condition.

47. $\frac{d\mathbf{r}}{dt} = \langle 1-2t, 4t\rangle, \quad \mathbf{r}(0) = \langle 3, 1\rangle$

48. $\mathbf{r}'(t) = \mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{k}$

49. $\mathbf{r}'(t) = t^2\mathbf{i} + 5t\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{j} + 2\mathbf{k}$

$\mathbf{r}'(t) = \langle \sin 3t, \sin 3t, t \rangle, \quad \mathbf{r}\left(\frac{\pi}{2}\right) = \left\langle 2, 4, \frac{\pi^2}{4} \right\rangle$

51. $\mathbf{r}''(t) = 16\mathbf{k}, \quad \mathbf{r}(0) = \langle 1, 0, 0 \rangle, \quad \mathbf{r}'(0) = \langle 0, 1, 0 \rangle$

52. $\mathbf{r}''(t) = \left\langle e^{2t-2}, t^2 - 1, 1 \right\rangle, \quad \mathbf{r}(1) = \langle 0, 0, 1 \rangle,$
$\mathbf{r}'(1) = \langle 2, 0, 0 \rangle$

53. $\mathbf{r}''(t) = \langle 0, 2, 0 \rangle, \quad \mathbf{r}(3) = \langle 1, 1, 0 \rangle,$
$\mathbf{r}'(3) = \langle 0, 0, 1 \rangle$

54. $\mathbf{r}''(t) = \left\langle e^t, \sin t, \cos t \right\rangle, \quad \mathbf{r}(0) = \langle 1, 0, 1 \rangle,$
$\mathbf{r}'(0) = \langle 0, 2, 2 \rangle$

55. Find the location at $t = 3$ of a particle whose path (Figure 8) satisfies

$$\frac{d\mathbf{r}}{dt} = \left\langle 2t - \frac{1}{(t+1)^2}, 2t - 4 \right\rangle, \qquad \mathbf{r}(0) = \langle 3, 8 \rangle$$

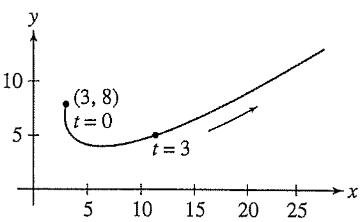

FIGURE 8 Particle path.

56. Find the location and velocity at $t = 4$ of a particle whose path satisfies

$$\frac{d\mathbf{r}}{dt} = \left\langle 2t^{-1/2}, 6, 8t \right\rangle, \qquad \mathbf{r}(1) = \langle 4, 9, 2 \rangle$$

57. A fighter plane, which can shoot a laser beam straight ahead, travels along the path $\mathbf{r}(t) = \left\langle 5 - t, 21 - t^2, 3 - t^3/27 \right\rangle$. Show that there is precisely one time t at which the pilot can hit a target located at the origin.

58. The fighter plane of Exercise 57 travels along the path $\mathbf{r}(t) = \left\langle t - t^3, 12 - t^2, 3 - t \right\rangle$. Show that the pilot cannot hit any target on the x-axis.

59. Find all solutions to $\mathbf{r}'(t) = \mathbf{v}$ with initial condition $\mathbf{r}(1) = \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are constant vectors in $\mathbf{R}^3$.

60. Let $\mathbf{u}$ be a constant vector in $\mathbf{R}^3$. Find the solution of the equation $\mathbf{r}'(t) = (\sin t)\mathbf{u}$ satisfying $\mathbf{r}'(0) = \mathbf{0}$.

61. Find all solutions to $\mathbf{r}'(t) = 2\mathbf{r}(t)$, where $\mathbf{r}(t)$ is a vector-valued function in 3-space.

62. Show that $\mathbf{w}(t) = \langle \sin(3t + 4), \sin(3t - 2), \cos 3t \rangle$ satisfies the differential equation $\mathbf{w}''(t) = -9\mathbf{w}(t)$.

63. Prove that the **Bernoulli spiral** (Figure 9) with parametrization $\mathbf{r}(t) = \left\langle e^t \cos 4t, e^t \sin 4t \right\rangle$ has the property that the angle ψ between the position vector and the tangent vector is constant. Find the angle ψ in degrees.

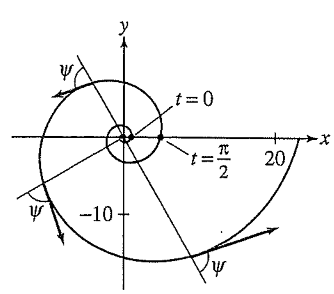

FIGURE 9 Bernoulli spiral.

64. A curve in polar form $r = f(\theta)$ has parametrization

$$\mathbf{r}(\theta) = f(\theta) \langle \cos\theta, \sin\theta \rangle$$

Let ψ be the angle between the radial and tangent vectors (Figure 10). Prove that

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{f(\theta)}{f'(\theta)}$$

Hint: Compute $\mathbf{r}(\theta) \times \mathbf{r}'(\theta)$ and $\mathbf{r}(\theta) \cdot \mathbf{r}'(\theta)$.

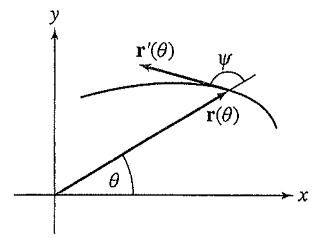

FIGURE 10 Curve with polar parametrization $\mathbf{r}(\theta) = f(\theta) \langle \cos\theta, \sin\theta \rangle$.

65. Prove that if $\|\mathbf{r}(t)\|$ takes on a local minimum or maximum value at t_0, then $\mathbf{r}(t_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Explain how this result is related to Figure 11. *Hint:* Observe that if $\|\mathbf{r}(t_0)\|$ is a minimum, then $\mathbf{r}(t)$ is tangent at t_0 to the sphere of radius $\|\mathbf{r}(t_0)\|$ centered at the origin.

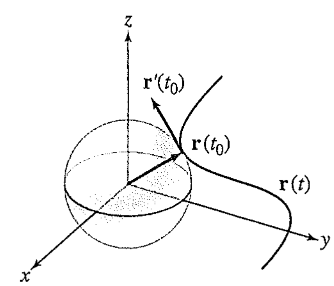

FIGURE 11

66. Newton's Second Law of Motion in vector form states that $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, where $\mathbf{F}$ is the force acting on an object of mass m and $\mathbf{p} = m\mathbf{r}'(t)$ is the object's momentum. The analogs of force and momentum for rotational motion are the **torque** $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ and **angular momentum**

$$\mathbf{J} = \mathbf{r}(t) \times \mathbf{p}(t)$$

Use the Second Law to prove that $\boldsymbol{\tau} = \dfrac{d\mathbf{J}}{dt}$.

13.3 SUMMARY

- The length s of a path $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$ is

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

- Arc length function: $s(t) = \displaystyle\int_a^t \|\mathbf{r}'(u)\|\,du$
- Speed is the derivative of distance traveled with respect to time:

$$v(t) = \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

- The velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$ points in the direction of motion [provided that $\mathbf{r}'(t) \neq \mathbf{0}$] and its magnitude $v(t) = \|\mathbf{r}'(t)\|$ is the object's speed.
- We say that $\mathbf{r}(s)$ is an *arc length parametrization* if $\|\mathbf{r}'(s)\| = 1$ for all s. In this case, the length of the path for $a \le s \le b$ is $b - a$.
- If $\mathbf{r}(t)$ is any parametrization such that $\mathbf{r}'(t) \neq \mathbf{0}$ for all t, then

$$\mathbf{r}_1(s) = \mathbf{r}(g^{-1}(s))$$

is an arc length parametrization, where $t = g^{-1}(s)$ is the inverse of the arc length function $s = g(t)$.

13.3 EXERCISES

Preliminary Questions

1. At a given instant, a car on a roller coaster has velocity vector $\mathbf{r}' = \langle 25, -35, 10\rangle$ (in miles per hour). What would the velocity vector be if the speed were doubled? What would it be if the car's direction were reversed but its speed remained unchanged?

2. Two cars travel in the same direction along the same roller coaster (at different times). Which of the following statements about their velocity vectors at a given point P on the roller coaster are true?

(a) The velocity vectors are identical.

(b) The velocity vectors point in the same direction but may have different lengths.

(c) The velocity vectors may point in opposite directions.

3. A mosquito flies along a parabola with speed $v(t) = t^2$. Let $L(t)$ be the total distance traveled at time t.

(a) How fast is $L(t)$ changing at $t = 2$?

(b) Is $L(t)$ equal to the mosquito's distance from the origin?

4. What is the length of the path traced by $\mathbf{r}(t)$ for $4 \le t \le 10$ if $\mathbf{r}(t)$ is an arc length parametrization?

Exercises

In Exercises 1–8, compute the length of the curve over the given interval.

1. $\mathbf{r}(t) = \langle 3t, 4t - 3, 6t + 1\rangle, \quad 0 \le t \le 3$

2. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{k}, \quad 11 \le t \le 15$

3. $\mathbf{r}(t) = \langle 2t, \ln t, t^2\rangle, \quad 1 \le t \le 4$

4. $\mathbf{r}(t) = \langle \cos t, \sin t, t^{3/2}\rangle, \quad 0 \le t \le 2\pi$

5. $\mathbf{r}(t) = \langle t, 4t^{3/2}, 2t^{3/2}\rangle, \quad 0 \le t \le 3$

6. $\mathbf{r}(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3\rangle, \quad 0 \le t \le 2$

7. $\mathbf{r}(t) = \langle t\cos t, t\sin t, 3t\rangle, \quad 0 \le t \le 2\pi$ *Hint:*

$$\int \sqrt{t^2 + a^2}\,dt = \frac{1}{2}t\sqrt{t^2 + a^2} + \frac{1}{2}a^2 \ln\left(t + \sqrt{t^2 + a^2}\right)$$

8. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (t^2 - 3)\mathbf{k}, \quad 0 \le t \le 2$ (See the hint above.)

In Exercises 9 and 10, compute the arc length function $s(t) = \displaystyle\int_a^t \|\mathbf{r}'(u)\|\,du$ *for the given value of a.*

9. $\mathbf{r}(t) = \langle t^2, 2t^2, t^3\rangle, \quad a = 0$

10. $\mathbf{r}(t) = \langle 4t^{1/2}, \ln t, 2t\rangle, \quad a = 1$

In Exercises 11–16, find the speed at the given value of t.

11. $\mathbf{r}(t) = \langle 2t + 3, 4t - 3, 5 - t\rangle, \quad t = 4$

12. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle, \quad t = 1$

13. $\mathbf{r}(t) = \langle t, \ln t, (\ln t)^2\rangle, \quad t = 1$

14. $\mathbf{r}(t) = \langle e^{t-3}, 12, 3t^{-1}\rangle, \quad t = 3$

15. $\mathbf{r}(t) = \langle \sin 3t, \cos 4t, \cos 5t\rangle, \quad t = \frac{\pi}{2}$

16. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t\rangle, \quad t = 0$

17. At an air show, a jet has a trajectory following the curve $y = x^2$. If when the jet is at the point $(1, 1)$, it has a speed of 500 km/h, determine its tangent vector at this point.

18. What is the velocity vector of a particle traveling to the right along the hyperbola $y = x^{-1}$ with constant speed 5 cm/s when the particle's location is $\left(2, \frac{1}{2}\right)$?

19. A bee with velocity vector $\mathbf{r}'(t)$ starts out at the origin at $t = 0$ and flies around for T seconds. Where is the bee located at time T if $\int_0^T \mathbf{r}'(u)\,du = \mathbf{0}$? What does the quantity $\int_0^T \|\mathbf{r}'(u)\|\,du$ represent?

20. The DNA molecule comes in the form of a double helix, meaning two helices that wrap around one another. Suppose a single one of the helices has a radius of 10Å (1 angstrom Å $= 10^{-8}$ cm) and one full turn of the helix has a height of 34Å.

(a) Show that the helix can be parametrized by $\mathbf{r}(t) = \left\langle 10\cos t, 10\sin t, \frac{34t}{2\pi}\right\rangle$.

(b) Find the arc length of one full turn of the helix.

21. Let

$$\mathbf{r}(t) = \left\langle R\cos\left(\frac{2\pi Nt}{h}\right), R\sin\left(\frac{2\pi Nt}{h}\right), t\right\rangle, \qquad 0 \le t \le h$$

(a) Show that $\mathbf{r}(t)$ parametrizes a helix of radius R and height h making N complete turns.

(b) Guess which of the two springs in Figure 5 uses more wire.

(c) Compute the lengths of the two springs and compare.

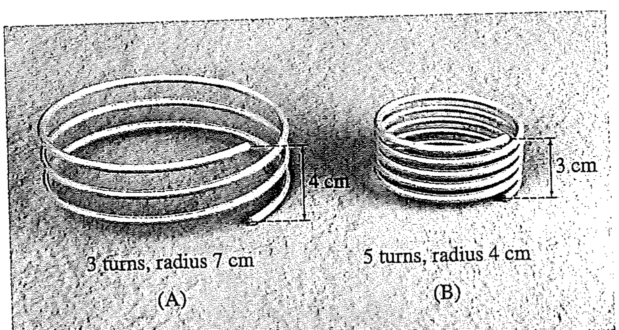

FIGURE 5 Which spring uses more wire?

22. Use Exercise 21 to find a general formula for the length of a helix of radius R and height h that makes N complete turns.

23. The cycloid generated by the unit circle has parametrization

$$\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$$

(a) Find the value of t in $[0, 2\pi]$ where the speed is at a maximum.

(b) Show that one arch of the cycloid has length 8. Recall the identity $\sin^2(t/2) = (1 - \cos t)/2$.

24. Which of the following is an arc length parametrization of a circle of radius 4 centered at the origin?

(a) $\mathbf{r}_1(t) = \langle 4\sin t, 4\cos t\rangle$

(b) $\mathbf{r}_2(t) = \langle 4\sin 4t, 4\cos 4t\rangle$

(c) $\mathbf{r}_3(t) = \left\langle 4\sin\frac{t}{4}, 4\cos\frac{t}{4}\right\rangle$

25. Let $\mathbf{r}(t) = \langle 3t + 1, 4t - 5, 2t\rangle$.

(a) Evaluate the arc length integral $s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du$.

(b) Find the inverse $g(s)$ of $s(t)$.

(c) Verify that $\mathbf{r}_1(s) = \mathbf{r}(g(s))$ is an arc length parametrization.

26. Find an arc length parametrization of the line $y = 4x + 9$.

27. Let $\mathbf{r}(t) = \mathbf{w} + t\mathbf{v}$ be the parametrization of a line.

(a) Show that the arc length function $s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du$ is given by $s(t) = t\|\mathbf{v}\|$. This shows that $\mathbf{r}(t)$ is an arc length parametrizaton if and only if $\mathbf{v}$ is a unit vector.

(b) Find an arc length parametrization of the line with $\mathbf{w} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 3, 4, 5\rangle$.

28. Find an arc length parametrization of the circle in the plane $z = 9$ with radius 4 and center $(1, 4, 9)$.

29. Find a path that traces the circle in the plane $y = 10$ with radius 4 and center $(2, 10, -3)$ with constant speed 8.

30. Find an arc length parametrization of the curve $\mathbf{r}(t) = \left\langle t, \frac{2}{3}t^{3/2}, \frac{2}{\sqrt{3}}t^{3/2}\right\rangle$, with the parameter s measuring from $(0, 0, 0)$.

31. Find an arc length parametrization of the curve $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{2}{3}t^{3/2}\right\rangle$, with the parameter s measuring from $(1, 0, 0)$.

32. Find an arc length parametrization of $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^t\rangle$.

33. Find an arc length parametrization of $\mathbf{r}(t) = \langle t^2, t^3\rangle$.

34. Find an arc length parametrization of the cycloid with parametrization $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

35. Find an arc length parametrization of the line $y = mx$ for an arbitrary slope m.

36. Express the arc length L of $y = x^3$ for $0 \le x \le 8$ as an integral in two ways, using the parametrizations $\mathbf{r}_1(t) = \langle t, t^3\rangle$ and $\mathbf{r}_2(t) = \langle t^3, t^9\rangle$. Do not evaluate the integrals, but use substitution to show that they yield the same result.

37. The curve known as the **Bernoulli spiral** (Figure 6) has parametrization $\mathbf{r}(t) = \langle e^t \cos 4t, e^t \sin 4t\rangle$.

(a) Evaluate $s(t) = \int_{-\infty}^t \|\mathbf{r}'(u)\|\,du$. It is convenient to take lower limit $-\infty$ because $\mathbf{r}(-\infty) = \langle 0, 0\rangle$.

(b) Use (a) to find an arc length parametrization of $\mathbf{r}(t)$.

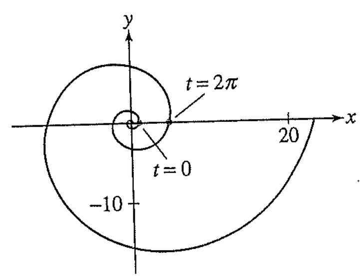

FIGURE 6 Bernoulli spiral.

13.4 SUMMARY

- A parametrization $\mathbf{r}(t)$ is called *regular* if $\mathbf{r}'(t) \neq \mathbf{0}$ for all t. If $\mathbf{r}(t)$ is regular, we define the *unit tangent vector* $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.
- *Curvature* is defined by $\kappa(s) = \left\| \dfrac{d\mathbf{T}}{ds} \right\|$, where $\mathbf{r}(s)$ is an arc length parametrization or $\kappa(s) = \dfrac{1}{v(t)} \left\| \dfrac{d\mathbf{T}}{dt} \right\|$ if $\mathbf{r}(t)$ is not an arc length parametrization
- In practice, we compute curvature using the following formula, which is valid for arbitrary regular parametrizations:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

- The curvature at a point on a graph $y = f(x)$ in the plane is

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}$$

- If $\|\mathbf{T}'(t)\| \neq 0$, we define the *unit normal vector* $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.
- $\mathbf{T}'(t) = \kappa(t)v(t)\mathbf{N}(t)$
- The *binormal vector* is defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.
- The *osculating plane* at a point P on a curve C is the plane through P determined by the vectors $\mathbf{T}_P$ and $\mathbf{N}_P$. It has normal vector $\mathbf{B}_P$. It is defined only if the curvature κ_P at P is nonzero.
- The *osculating circle* Osc_P is the circle in the osculating plane through P of radius $R = 1/\kappa_P$ whose center Q lies in the normal direction $\mathbf{N}_P$:

$$\overrightarrow{OQ} = \mathbf{r}(t_0) + \kappa_P^{-1}\mathbf{N}_P = \mathbf{r}(t_0) + R\mathbf{N}_P$$

The center of Osc_P is called the *center of curvature* and R the *radius of curvature*.

13.4 EXERCISES

Preliminary Questions

1. What is the unit tangent vector of a line with direction vector $\mathbf{v} = \langle 2, 1, -2 \rangle$?
2. What is the curvature of a circle of radius 4?
3. Which has larger curvature, a circle of radius 2 or a circle of radius 4?
4. What is the curvature of $\mathbf{r}(t) = \langle 2 + 3t, 7t, 5 - t \rangle$?
5. What is the curvature at a point where $\mathbf{T}'(s) = \langle 1, 2, 3 \rangle$ in an arc length parametrization $\mathbf{r}(s)$?
6. What is the radius of curvature of a circle of radius 4?
7. What is the radius of curvature at P if $\kappa_P = 9$?

Exercises

In Exercises 1–6, calculate $\mathbf{r}'(t)$ *and* $\mathbf{T}(t)$*, and evaluate* $\mathbf{T}(1)$.

1. $\mathbf{r}(t) = \langle 4t^2, 9t \rangle$
2. $\mathbf{r}(t) = \langle e^t, t^2 \rangle$
3. $\mathbf{r}(t) = \langle 3 + 4t, 3 - 5t, 9t \rangle$
4. $\mathbf{r}(t) = \langle 1 + 2t, t^2, 3 - t^2 \rangle$
5. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$
6. $\mathbf{r}(t) = \langle e^t, e^{-t}, t^2 \rangle$

In Exercises 7–10, use Eq. (3) to calculate the curvature function $\kappa(t)$.

7. $\mathbf{r}(t) = \langle 1, e^t, t \rangle$
8. $\mathbf{r}(t) = \langle 4\cos t, t, 4\sin t \rangle$
9. $\mathbf{r}(t) = \langle 4t + 1, 4t - 3, 2t \rangle$
10. $\mathbf{r}(t) = \langle t^{-1}, 1, t \rangle$

In Exercises 11–14, use Eq. (3) to evaluate the curvature at the given point.

11. $\mathbf{r}(t) = \langle 1/t, 1/t^2, t^2 \rangle$, $t = -1$
12. $\mathbf{r}(t) = \langle 3 - t, e^{t-4}, 8t - t^2 \rangle$, $t = 4$
13. $\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle$, $t = \frac{\pi}{2}$
14. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t \rangle$, $t = 0$

In Exercises 15–18, find the curvature of the plane curve at the point indicated.

15. $y = e^t$, $t = 3$
16. $y = \cos x$, $x = 0$

17. $y = t^4, \quad t = 2$

18. $y = t^n, \quad t = 1$

19. Find the curvature of $\mathbf{r}(t) = \langle 2\sin t, \cos 3t, t\rangle$ at $t = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$ (Figure 17).

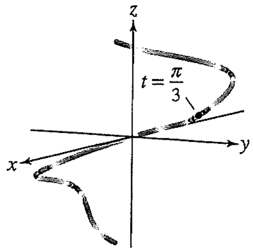

FIGURE 17 The curve $\mathbf{r}(t) = \langle 2\sin t, \cos 3t, t\rangle$.

20. GU Find the curvature function $\kappa(x)$ for $y = \sin x$. Use a computer algebra system to plot $\kappa(x)$ for $0 \le x \le 2\pi$. Prove that the curvature takes its maximum at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. *Hint:* As a shortcut to finding the max, observe that the maximum of the numerator and the minimum of the denominator of $\kappa(x)$ occur at the same points.

21. Show that the tractrix $\mathbf{r}(t) = \langle t - \tanh t, \operatorname{sech} t\rangle$ has the curvature function $\kappa(t) = \operatorname{sech} t$.

22. Show that curvature at an inflection point of a plane curve $y = f(x)$ is zero.

23. Find the value(s) of α such that the curvature of $y = e^{\alpha x}$ at $x = 0$ is as large as possible.

24. Find the point of maximum curvature on $y = e^x$.

25. Show that the curvature function of the parametrization $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$ of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is

$$\kappa(t) = \frac{ab}{(b^2\cos^2 t + a^2\sin^2 t)^{3/2}} \qquad \boxed{10}$$

26. Use a sketch to predict where the points of minimal and maximal curvature occur on an ellipse. Then use Eq. (10) to confirm or refute your prediction.

27. In the notation of Exercise 25, assume that $a \ge b$. Show that $b/a^2 \le \kappa(t) \le a/b^2$ for all t.

28. Use Eq. (3) to prove that for a plane curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$,

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} \qquad \boxed{11}$$

In Exercises 29–32, use Eq. (11) to compute the curvature at the given point.

29. $\langle t^2, t^3\rangle, \quad t = 2$

30. $\langle \cosh s, s\rangle, \quad s = 0$

31. $\langle t\cos t, \sin t\rangle, \quad t = \pi$

32. $\langle \sin 3s, 2\sin 4s\rangle, \quad s = \frac{\pi}{2}$

33. Let $s(t) = \int_{-\infty}^{t} \|\mathbf{r}'(u)\|\,du$ for the Bernoulli spiral $\mathbf{r}(t) = \langle e^t\cos 4t, e^t\sin 4t\rangle$ (see Exercise 37 in Section 13.3). Show that the radius of curvature is proportional to $s(t)$.

34. The **Cornu spiral** is the plane curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, where

$$x(t) = \int_0^t \sin\frac{u^2}{2}\,du, \qquad y(t) = \int_0^t \cos\frac{u^2}{2}\,du$$

Verify that $\kappa(t) = |t|$. Since the curvature increases linearly, the Cornu spiral is used in highway design to create transitions between straight and curved road segments (Figure 18).

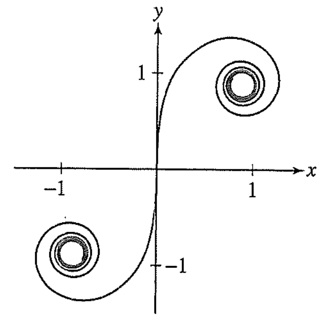

FIGURE 18 Cornu spiral.

35. CAS Plot the **clothoid** $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, and compute its curvature $\kappa(t)$ where

$$x(t) = \int_0^t \sin\frac{u^3}{3}\,du, \qquad y(t) = \int_0^t \cos\frac{u^3}{3}\,du$$

36. Find the normal vector $\mathbf{N}(\theta)$ to $\mathbf{r}(\theta) = R\langle\cos\theta, \sin\theta\rangle$, the circle of radius R. Does $\mathbf{N}(\theta)$ point inside or outside the circle? Draw $\mathbf{N}(\theta)$ at $\theta = \frac{\pi}{4}$ with $R = 4$.

37. Find the normal vector $\mathbf{N}(t)$ to $\mathbf{r}(t) = \langle 4, \sin 2t, \cos 2t\rangle$.

38. Sketch the graph of $\mathbf{r}(t) = \langle t, t^3\rangle$. Since $\mathbf{r}'(t) = \langle 1, 3t^2\rangle$, the unit normal $\mathbf{N}(t)$ points in one of the two directions $\pm\langle -3t^2, 1\rangle$. Which sign is correct at $t = 1$? Which is correct at $t = -1$?

39. Find the normal vectors to $\mathbf{r}(t) = \langle t, \cos t\rangle$ at $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$.

40. Find the normal vector to the Cornu spiral (Exercise 34) at $t = \sqrt{\pi}$.

In Exercises 41–44, find **T**, **N**, *and* **B** *for the curve at the indicated point. Hint: After finding* **T**′, *plug in the specific value for t before computing* **N** *and* **B**.

41. $\mathbf{r}(t) = \langle 0, t, t^2\rangle$ at $(0, 1, 1)$. In this case, draw the curve and the three resultant vectors in 3-space.

42. $\mathbf{r}(t) = \langle\cos t, \sin t, 2\rangle$ at $(1, 0, 2)$. In this case, draw the curve and the three resultant vectors in 3-space.

43. $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3\rangle$ at $(1, 1, \frac{2}{3})$.

44. $\mathbf{r}(t) = \langle t, t, e^t\rangle$ at $(0, 0, 1)$.

45. Find the normal vector to the clothoid (Exercise 35) at $t = \pi^{1/3}$.

46. **Method for Computing N** Let $v(t) = \|\mathbf{r}'(t)\|$. Show that

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|} \qquad \boxed{12}$$

Hint: **N** is the unit vector in the direction $\mathbf{T}'(t)$. Differentiate $\mathbf{T}(t) = \mathbf{r}'(t)/v(t)$ to show that $v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)$ is a positive multiple of $\mathbf{T}'(t)$.

In Exercises 47–52, use Eq. (12) to find **N** *at the point indicated.*

47. $\langle t^2, t^3\rangle, \quad t = 1$

48. $\langle t - \sin t, 1 - \cos t\rangle, \quad t = \pi$

49. $\langle t^2/2, t^3/3, t\rangle, \quad t = 1$

50. $\langle t^{-1}, t, t^2\rangle, \quad t = -1$

51. $\langle t, e^t, t\rangle, \quad t = 0$

52. $\langle \cosh t, \sinh t, t^2\rangle, \quad t = 0$

53. Let $\mathbf{r}(t) = \left\langle t, \frac{4}{3}t^{3/2}, t^2\right\rangle$.

(a) Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point corresponding to $t = 1$.

(b) Find the equation of the osculating plane at the point corresponding to $t = 1$.

54. Let $\mathbf{r}(t) = \langle \cos t, \sin t, \ln(\cos t)\rangle$.

(a) Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at $(1, 0, 0)$.

(b) Find the equation of the osculating plane at $(1, 0, 0)$.

55. Let $\mathbf{r}(t) = \left\langle t, 1 - t, t^2\right\rangle$.

(a) Find the general formulas for $\mathbf{T}$ and $\mathbf{N}$ as functions of t.

(b) Find the general formula for $\mathbf{B}$ as a function of t.

(c) What can you conclude about the osculating planes of the curve based on your answer to b?

56. **(a)** What does it mean for a space curve to have a constant unit tangent vector $\mathbf{T}$?

(b) What does it mean for a space curve to have a constant normal vector $\mathbf{N}$?

(c) What does it mean for a space curve to have a constant binormal vector $\mathbf{B}$?

57. Let $f(x) = x^2$. Show that the center of the osculating circle at (x_0, x_0^2) is given by $\left(-4x_0^3, \frac{1}{2} + 3x_0^2\right)$.

58. Use Eq. (9) to find the center of curvature to $\mathbf{r}(t) = \langle t^2, t^3\rangle$ at $t = 1$.

In Exercises 59–66, find a parametrization of the osculating circle at the point indicated.

59. $\mathbf{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \frac{\pi}{4}$

60. $\mathbf{r}(t) = \langle \sin t, \cos t\rangle, \quad t = 0$

61. $y = x^2, \quad x = 1$

62. $y = \sin x, \quad x = \frac{\pi}{2}$

63. $\langle t - \sin t, 1 - \cos t\rangle, \quad t = \pi$

64. $\mathbf{r}(t) = \langle t^2/2, t^3/3, t\rangle, \quad t = 0$

65. $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle, \quad t = 0$

66. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t\rangle, \quad t = 0$

67. Figure 19 shows the graph of the half-ellipse $y = \pm\sqrt{2rx - px^2}$, where r and p are positive constants. Show that the radius of curvature at the origin is equal to r. *Hint:* One way of proceeding is to write the ellipse in the form of Exercise 25 and apply Eq. (10).

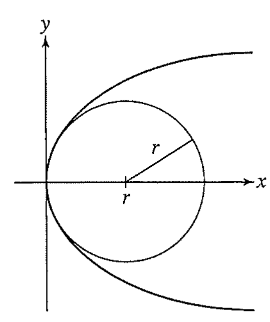

FIGURE 19 The curve $y = \pm\sqrt{2rx - px^2}$ and the osculating circle at the origin.

68. In a recent study of laser eye surgery by Gatinel, Hoang-Xu and Azar, a vertical cross section of the cornea is modeled by the half-ellipse of Exercise 67. Show that the half-ellipse can be written in the form $x = f(y)$, where $f(y) = p^{-1}\left(r - \sqrt{r^2 - py^2}\right)$. During surgery, tissue is removed to a depth $t(y)$ at height y for $-S \le y \le S$, where $t(y)$ is given by Munnerlyn's equation (for some $R > r$):

$$t(y) = \sqrt{R^2 - S^2} - \sqrt{R^2 - y^2} - \sqrt{r^2 - S^2} + \sqrt{r^2 - y^2}$$

After surgery, the cross section of the cornea has the shape $x = f(y) + t(y)$ (Figure 20). Show that after surgery, the radius of curvature at the point P (where $y = 0$) is R.

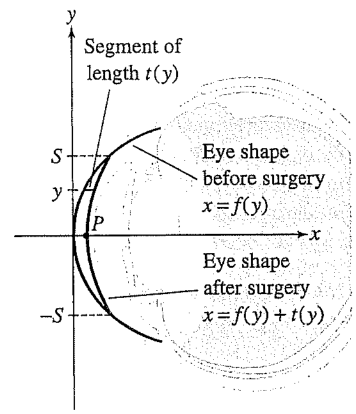

FIGURE 20 Contour of cornea before and after surgery.

69. The **angle of inclination** at a point P on a plane curve is the angle θ between the unit tangent vector $\mathbf{T}$ and the x-axis (Figure 21). Assume that $\mathbf{r}(s)$ is a arc length parametrization, and let $\theta = \theta(s)$ be the angle of inclination at $\mathbf{r}(s)$. Prove that

$$\kappa(s) = \left|\frac{d\theta}{ds}\right|$$ 13

Hint: Observe that $\mathbf{T}(s) = \langle \cos\theta(s), \sin\theta(s)\rangle$.

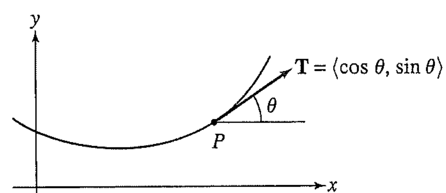

FIGURE 21 The curvature at P is the quantity $|d\theta/ds|$.

70. A particle moves along the path $y = x^3$ with unit speed. How fast is the tangent turning (i.e., how fast is the angle of inclination changing) when the particle passes through the point $(2, 8)$?

71. Let $\theta(x)$ be the angle of inclination at a point on the graph $y = f(x)$ (see Exercise 69).

(a) Use the relation $f'(x) = \tan\theta$ to prove that $\dfrac{d\theta}{dx} = \dfrac{f''(x)}{(1 + f'(x)^2)}$.

(b) Use the arc length integral to show that $\dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$.

(c) Now give a proof of Eq. (5) using Eq. (13).

72. Use the parametrization $\mathbf{r}(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$ to show that a curve $r = f(\theta)$ in polar coordinates has curvature

$$\kappa(\theta) = \frac{|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)|}{(f(\theta)^2 + f'(\theta)^2)^{3/2}}$$ 14

13.5 EXERCISES

Preliminary Questions

1. If a particle travels with constant speed, must its acceleration vector be zero? Explain.

2. For a particle in uniform circular motion around a circle, which of the vectors $\mathbf{v}(t)$ or $\mathbf{a}(t)$ always points toward the center of the circle?

3. Two objects travel to the right along the parabola $y = x^2$ with nonzero speed. Which of the following statements must be true?
(a) Their velocity vectors point in the same direction.
(b) Their velocity vectors have the same length.
(c) Their acceleration vectors point in the same direction.

4. Use the decomposition of acceleration into tangential and normal components to explain the following statement: If the speed is constant, then the acceleration and velocity vectors are orthogonal.

5. If a particle travels along a straight line, then the acceleration and velocity vectors are (choose the correct description):
(a) orthogonal. (b) parallel.

6. What is the length of the acceleration vector of a particle traveling around a circle of radius 2 cm with constant velocity 4 cm/s?

7. Two cars are racing around a circular track. If, at a certain moment, both of their speedometers read 110 mph. then the two cars have the same (choose one):
(a) $a_{\mathbf{T}}$ (b) $a_{\mathbf{N}}$

Exercises

1. Use the table below to calculate the difference quotients $\dfrac{\mathbf{r}(1+h)-\mathbf{r}(1)}{h}$ for $h = -0.2, -0.1, 0.1, 0.2$. Then estimate the velocity and speed at $t = 1$.

$\mathbf{r}(0.8)$	$\langle 1.557, 2.459, -1.970\rangle$
$\mathbf{r}(0.9)$	$\langle 1.559, 2.634, -1.740\rangle$
$\mathbf{r}(1)$	$\langle 1.540, 2.841, -1.443\rangle$
$\mathbf{r}(1.1)$	$\langle 1.499, 3.078, -1.035\rangle$
$\mathbf{r}(1.2)$	$\langle 1.435, 3.342, -0.428\rangle$

2. Draw the vectors $\mathbf{r}(2+h) - \mathbf{r}(2)$ and $\dfrac{\mathbf{r}(2+h)-\mathbf{r}(2)}{h}$ for $h = 0.5$ for the path in Figure 10. Draw $\mathbf{v}(2)$ (using a rough estimate for its length).

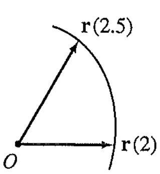

FIGURE 10

In Exercises 3–6, calculate the velocity and acceleration vectors and the speed at the time indicated.

3. $\mathbf{r}(t) = \langle t^3, 1-t, 4t^2\rangle, \quad t = 1$

4. $\mathbf{r}(t) = e^t\mathbf{j} - \cos(2t)\mathbf{k}, \quad t = 0$

5. $\mathbf{r}(\theta) = \langle \sin\theta, \cos\theta, \cos 3\theta\rangle, \quad \theta = \frac{\pi}{3}$

6. $\mathbf{r}(s) = \left\langle \dfrac{1}{1+s^2}, \dfrac{s}{1+s^2}\right\rangle, \quad s = 2$

7. Find $\mathbf{a}(t)$ for a particle moving around a circle of radius 8 cm at a constant speed of $v = 4$ cm/s (see Example 5). Draw the path and acceleration vector at $t = \frac{\pi}{4}$.

8. Sketch the path $\mathbf{r}(t) = \langle 1-t^2, 1-t\rangle$ for $-2 \le t \le 2$, indicating the direction of motion. Draw the velocity and acceleration vectors at $t = 0$ and $t = 1$.

9. Sketch the path $\mathbf{r}(t) = \langle t^2, t^3\rangle$ together with the velocity and acceleration vectors at $t = 1$.

10. The paths $\mathbf{r}(t) = \langle t^2, t^3\rangle$ and $\mathbf{r}_1(t) = \langle t^4, t^6\rangle$ trace the same curve, and $\mathbf{r}_1(1) = \mathbf{r}(1)$. Do you expect either the velocity vectors or the acceleration vectors of these paths at $t = 1$ to point in the same direction? Compute these vectors and draw them on a single plot of the curve.

In Exercises 11–14, find $\mathbf{v}(t)$ given $\mathbf{a}(t)$ and the initial velocity.

11. $\mathbf{a}(t) = \langle t, 4\rangle, \quad \mathbf{v}(0) = \langle \frac{1}{3}, -2\rangle$

12. $\mathbf{a}(t) = \langle e^t, 0, t+1\rangle, \quad \mathbf{v}(0) = \langle 1, -3, \sqrt{2}\rangle$

13. $\mathbf{a}(t) = \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}$

14. $\mathbf{a}(t) = t^2\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j}$

In Exercises 15–18, find $\mathbf{r}(t)$ and $\mathbf{v}(t)$ given $\mathbf{a}(t)$ and the initial velocity and position.

15. $\mathbf{a}(t) = \langle t, 4\rangle, \quad \mathbf{v}(0) = \langle 3, -2\rangle, \quad \mathbf{r}(0) = \langle 0, 0\rangle$

16. $\mathbf{a}(t) = \langle e^t, 2t, t+1\rangle, \quad \mathbf{v}(0) = \langle 1, 0, 1\rangle, \quad \mathbf{r}(0) = \langle 2, 1, 1\rangle$

17. $\mathbf{a}(t) = t\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}$

18. $\mathbf{a}(t) = \cos t\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{i}$

In Exercises 19–26, recall that $g = 9.8$ m/s^2 = 32 ft/s^2 is the acceleration due to gravity on the earth's surface.

19. A bullet is fired from the ground at an angle of 45°. What initial speed must the bullet have in order to hit the top of a 120-m tower located 180 m away?

20. Find the initial velocity vector $\mathbf{v}_0$ of a projectile released with initial speed 100 m/s that reaches a maximum height of 300 m.

21. Show that a projectile fired at an angle θ with initial speed v_0 travels a total distance $(v_0^2/g)\sin 2\theta$ before hitting the ground. Conclude that the maximum distance (for a given v_0) is attained for $\theta = 45°$.

22. One player throws a baseball to another player standing 25 m away, with initial speed 18 m/s. Use the result of Exercise 21 to find two angles θ at which the ball can be released. Which angle gets the ball there faster?

23. A bullet is fired at an angle $\theta = \frac{\pi}{4}$ at a tower located $d = 600$ m away, with initial speed $v_0 = 120$ m/s. Find the height H at which the bullet hits the tower.

Show that a bullet fired at an angle θ will hit the top of an h-meter wer located d meters away if its initial speed is

$$v_0 = \frac{\sqrt{g/2}\, d \sec\theta}{\sqrt{d\tan\theta - h}}$$

25. In the Superbowl, a quarterback throws a football while standing at the very center of the field on the 50 yard line. The ball leaves his hand at a height of 5 ft and has initial velocity $\mathbf{v}_0 = 40\mathbf{i} + 35\mathbf{j} + 32\mathbf{k}$ ft/s. Assume an acceleration of 32 ft/s^2 due to gravity and that the $\mathbf{i}$ vector points down the field toward the endzone and the $\mathbf{j}$ vector points to the sideline. The field is 150 ft in width and 300 ft in length.

(a) Determine the position function that gives the position of the ball t seconds after it is thrown.

(b) The ball is caught by a player 5 ft above the ground. Is the player in bounds or out of bounds when he receives the ball? Assume the player is standing vertically with both toes on the ground at the time of reception.

26. In the Women's World Cup, Brazilian soccer star Marta has a penalty kick in the quarter-final match. She kicks the soccer ball from ground level with (x, y)-coordinates (85, 20) on the soccer field shown in Figure 11 and with an initial velocity $\mathbf{v}_0 = 10\mathbf{i} - 5\mathbf{j} + 25\mathbf{k}$ ft/s. Assume an acceleration of 32 ft/s^2 due to gravity and that the goal net has a height of 8 ft and a total width of 24 ft.

(a) Determine the position function that gives the position of the ball t seconds after it is hit.

(b) Does the ball go in the goal before hitting the ground? Explain why or why not.

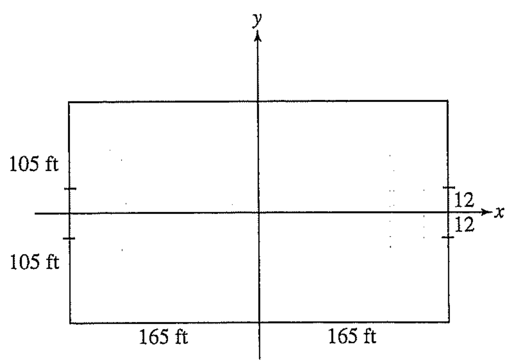

FIGURE 11

27. A constant force $\mathbf{F} = \langle 5, 2\rangle$ (in newtons) acts on a 10-kg mass. Find the position of the mass at $t = 10$ s if it is located at the origin at $t = 0$ and has initial velocity $\mathbf{v}_0 = \langle 2, -3\rangle$ (in meters per second).

28. A force $\mathbf{F} = \langle 24t, 16 - 8t\rangle$ (in newtons) acts on a 4-kg mass. Find the position of the mass at $t = 3$ s if it is located at (10, 12) at $t = 0$ and has zero initial velocity.

29. A particle follows a path $\mathbf{r}(t)$ for $0 \le t \le T$, beginning at the origin O. The vector $\overline{\mathbf{v}} = \frac{1}{T}\int_0^T \mathbf{r}'(t)\,dt$ is called the **average velocity** vector. Suppose that $\overline{\mathbf{v}} = \mathbf{0}$. Answer and explain the following:

(a) Where is the particle located at time T if $\overline{\mathbf{v}} = \mathbf{0}$?

(b) Is the particle's average speed necessarily equal to zero?

30. At a certain moment, a moving particle has velocity $\mathbf{v} = \langle 2, 2, -1\rangle$ nd $\mathbf{a} = \langle 0, 4, 3\rangle$. Find $\mathbf{T}$, $\mathbf{N}$, and the decomposition of $\mathbf{a}$ into tangential rmal components.

31. At a certain moment, a particle moving along a path has velocity $\mathbf{v} = \langle 12, 20, 20\rangle$ and acceleration $\mathbf{a} = \langle 2, 1, -3\rangle$. Is the particle speeding up or slowing down?

In Exercises 32–35, use Eq. (2) to find the coefficients $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as a function of t (or at the specified value of t).

32. $\mathbf{r}(t) = \langle t^2, t^3\rangle$

33. $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$

34. $\mathbf{r}(t) = \langle t^{-1}, \ln t, t^2\rangle, \quad t = 1$

35. $\mathbf{r}(t) = \langle e^{2t}, t, e^{-t}\rangle, \quad t = 0$

In Exercise 36–43, find the decomposition of $\mathbf{a}(t)$ into tangential and normal components at the point indicated, as in Example 7.

36. $\mathbf{r}(t) = \langle e^t, 1 - t\rangle, \quad t = 0$

37. $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, 1 - 3t\right\rangle, \quad t = -1$

38. $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\right\rangle, \quad t = 1$

39. $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\right\rangle, \quad t = 4$

40. $\mathbf{r}(t) = \langle 4 - t, t + 1, t^2\rangle, \quad t = 2$

41. $\mathbf{r}(t) = \langle t, e^t, te^t\rangle, \quad t = 0$

42. $\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, \theta\rangle, \quad \theta = 0$

43. $\mathbf{r}(t) = \langle t, \cos t, t\sin t\rangle, \quad t = \frac{\pi}{2}$

44. Let $\mathbf{r}(t) = \langle t^2, 4t - 3\rangle$. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$, and show that the decomposition of $\mathbf{a}(t)$ into tangential and normal components is

$$\mathbf{a}(t) = \left(\frac{2t}{\sqrt{t^2+4}}\right)\mathbf{T} + \left(\frac{4}{\sqrt{t^2+4}}\right)\mathbf{N}$$

45. Find the components $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ of the acceleration vector of a particle moving along a circular path of radius $R = 100$ cm with constant velocity $v_0 = 5$ cm/s.

46. In the notation of Example 6, find the acceleration vector for a person seated in a car at (a) the highest point of the Ferris wheel and (b) the two points level with the center of the wheel.

47. Suppose that the Ferris wheel in Example 6 is rotating clockwise and that the point P at angle 45° has acceleration vector $\mathbf{a} = \langle 0, -50\rangle$ m/min^2 pointing down, as in Figure 12. Determine the speed and tangential acceleration of the Ferris wheel.

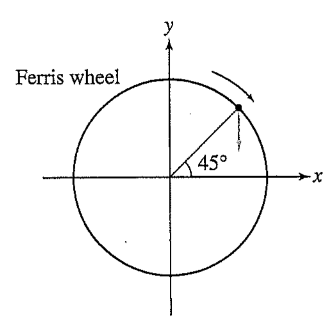

FIGURE 12

48. At time t_0, a moving particle has velocity vector $\mathbf{v} = 2\mathbf{i}$ and acceleration vector $\mathbf{a} = 3\mathbf{i} + 18\mathbf{k}$. Determine the curvature $\kappa(t_0)$ of the particle's path at time t_0.