- Standard basis vectors: $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.
- Every vector $\mathbf{v} = \langle a, b \rangle$ is a linear combination $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.
- Triangle Inequality: $\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

12.1 EXERCISES

Preliminary Questions

1. Answer true or false. Every nonzero vector is:

(a) equivalent to a vector based at the origin.

(b) equivalent to a unit vector based at the origin.

(c) parallel to a vector based at the origin.

(d) parallel to a unit vector based at the origin.

2. What is the length of $-3\mathbf{a}$ if $\|\mathbf{a}\| = 5$?

3. Suppose that $\mathbf{v}$ has components $\langle 3, 1 \rangle$. How, if at all, do the components change if you translate $\mathbf{v}$ horizontally 2 units to the left?

4. What are the components of the zero vector based at $P = (3, 5)$?

5. True or false?

(a) The vectors $\mathbf{v}$ and $-2\mathbf{v}$ are parallel.

(b) The vectors $\mathbf{v}$ and $-2\mathbf{v}$ point in the same direction.

6. Explain the commutativity of vector addition in terms of the Parallelogram Law.

Exercises

1. Sketch the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ with tail P and head Q, and compute their lengths. Are any two of these vectors equivalent?

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
P	(2, 4)	(−1, 3)	(−1, 3)	(4, 1)
Q	(4, 4)	(1, 3)	(2, 4)	(6, 3)

2. Sketch the vector $\mathbf{b} = \langle 3, 4 \rangle$ based at $P = (-2, -1)$.

3. What is the terminal point of the vector $\mathbf{a} = \langle 1, 3 \rangle$ based at $P = (2, 2)$? Sketch $\mathbf{a}$ and the vector $\mathbf{a}_0$ based at the origin and equivalent to $\mathbf{a}$.

4. Let $\mathbf{v} = \overrightarrow{PQ}$, where $P = (1, 1)$ and $Q = (2, 2)$. What is the head of the vector $\mathbf{v}'$ equivalent to $\mathbf{v}$ based at $(2, 4)$? What is the head of the vector $\mathbf{v}_0$ equivalent to $\mathbf{v}$ based at the origin? Sketch $\mathbf{v}$, $\mathbf{v}_0$, and $\mathbf{v}'$.

In Exercises 5–8, refer to Figure 21.

5. Find the components of $\mathbf{u}$.

6. Find the components of $\mathbf{v}$.

7. Find the components of $\mathbf{w}$.

8. Find the components of $\mathbf{q}$.

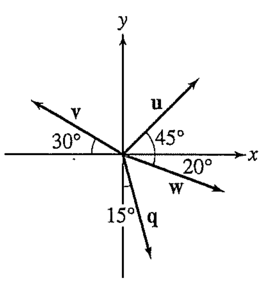

FIGURE 21

In Exercises 9–12, find the components of $\overrightarrow{PQ}$.

9. $P = (3, 2), \quad Q = (2, 7)$

10. $P = (1, -4), \quad Q = (3, 5)$

11. $P = (3, 5), \quad Q = (1, -4)$

12. $P = (0, 2), \quad Q = (5, 0)$

In Exercises 13–18, calculate.

13. $\langle 2, 1 \rangle + \langle 3, 4 \rangle$

14. $\langle -4, 6 \rangle - \langle 3, -2 \rangle$

15. $5\langle 6, 2 \rangle$

16. $4(\langle 1, 1 \rangle + \langle 3, 2 \rangle)$

17. $\left\langle -\frac{1}{2}, \frac{5}{3} \right\rangle + \left\langle 3, \frac{10}{3} \right\rangle$

18. $\langle \ln 2, e \rangle + \langle \ln 3, \pi \rangle$

19. Which of the vectors (A)–(C) in Figure 22 is equivalent to $\mathbf{v} - \mathbf{w}$?

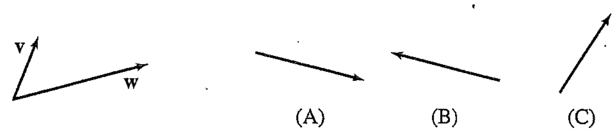

FIGURE 22

20. Sketch $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ for the vectors in Figure 23.

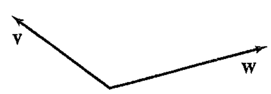

FIGURE 23

21. Sketch $2\mathbf{v}$, $-\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, and $2\mathbf{v} - \mathbf{w}$ for the vectors in Figure 24.

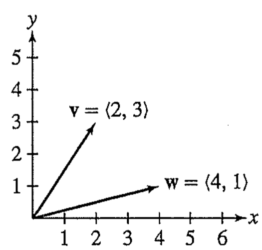

FIGURE 24

22. Sketch $\mathbf{v} = \langle 1, 3 \rangle$, $\mathbf{w} = \langle 2, -2 \rangle$, $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$.

23. Sketch $\mathbf{v} = \langle 0, 2 \rangle$, $\mathbf{w} = \langle -2, 4 \rangle$, $3\mathbf{v} + \mathbf{w}$, $2\mathbf{v} - 2\mathbf{w}$.

24. Sketch $\mathbf{v} = \langle -2, 1 \rangle$, $\mathbf{w} = \langle 2, 2 \rangle$, $\mathbf{v} + 2\mathbf{w}$, $\mathbf{v} - 2\mathbf{w}$.

25. Sketch the vector $\mathbf{v}$ such that $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ for $\mathbf{v}_1$ and $\mathbf{v}_2$ in Figure 25(A).

26. Sketch the vector sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$ in Figure 25(B).

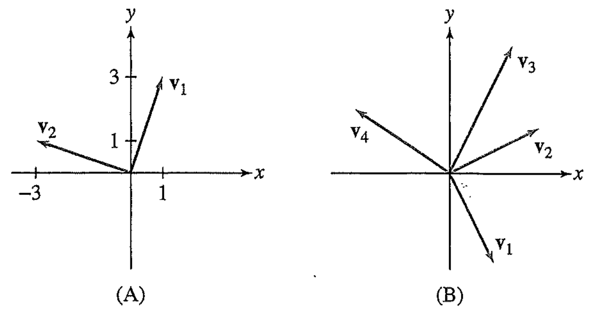

FIGURE 25

27. Let $\mathbf{v} = \overrightarrow{PQ}$, where $P = (-2, 5)$, $Q = (1, -2)$. Which of the following vectors with the given tails and heads are equivalent to $\mathbf{v}$?

(a) $(-3, 3)$, $(0, 4)$ **(b)** $(0, 0)$, $(3, -7)$
(c) $(-1, 2)$, $(2, -5)$ **(d)** $(4, -5)$, $(1, 4)$

28. Which of the following vectors are parallel to $\mathbf{v} = \langle 6, 9\rangle$ and which point in the same direction?

(a) $\langle 12, 18\rangle$ **(b)** $\langle 3, 2\rangle$ **(c)** $\langle 2, 3\rangle$
(d) $\langle -6, -9\rangle$ **(e)** $\langle -24, -27\rangle$ **(f)** $\langle -24, -36\rangle$

In Exercises 29–32, sketch the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$, and determine whether they are equivalent.

29. $A = (1, 1)$, $B = (3, 7)$, $P = (4, -1)$, $Q = (6, 5)$

30. $A = (1, 4)$, $B = (-6, 3)$, $P = (1, 4)$, $Q = (6, 3)$

31. $A = (-3, 2)$, $B = (0, 0)$, $P = (0, 0)$, $Q = (3, -2)$

32. $A = (5, 8)$, $B = (1, 8)$, $P = (1, 8)$, $Q = (-3, 8)$

In Exercises 33–36, are $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ parallel? And if so, do they point in the same direction?

33. $A = (1, 1)$, $B = (3, 4)$, $P = (1, 1)$, $Q = (7, 10)$

34. $A = (-3, 2)$, $B = (0, 0)$, $P = (0, 0)$, $Q = (3, 2)$

35. $A = (2, 2)$, $B = (-6, 3)$, $P = (9, 5)$, $Q = (17, 4)$

36. $A = (5, 8)$, $B = (2, 2)$, $P = (2, 2)$, $Q = (-3, 8)$

In Exercises 37–40, let $R = (-2, 7)$. Calculate the following:

37. The length of $\overrightarrow{OR}$

38. The components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2)$

39. The point P such that $\overrightarrow{PR}$ has components $\langle -2, 7\rangle$

40. The point Q such that $\overrightarrow{RQ}$ has components $\langle 8, -3\rangle$

In Exercises 41–48, find the given vector.

41. Unit vector $\mathbf{e}_\mathbf{v}$ where $\mathbf{v} = \langle 3, 4\rangle$

42. Unit vector $\mathbf{e}_\mathbf{w}$ where $\mathbf{w} = \langle 24, 7\rangle$

43. Vector of length 4 in the direction of $\mathbf{u} = \langle -1, -1\rangle$

44. Vector of length 3 in the direction of $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$

45. Vector of length 2 in the direction opposite to $\mathbf{v} = \mathbf{i} - \mathbf{j}$

46. Unit vector in the direction opposite to $\mathbf{v} = \langle -2, 4\rangle$

47. Unit vector $\mathbf{e}$ making an angle of $\frac{4\pi}{7}$ with the x-axis

48. Vector $\mathbf{v}$ of length 2 making an angle of 30° with the x-axis

49. Find all scalars λ such that $\lambda\langle 2, 3\rangle$ has length 1.

50. Find a vector $\mathbf{v}$ satisfying $3\mathbf{v} + \langle 5, 20\rangle = \langle 11, 17\rangle$.

51. What are the coordinates of the point P in the parallelogram in Figure 26(A)?

52. What are the coordinates a and b in the parallelogram in Figure 26(B)?

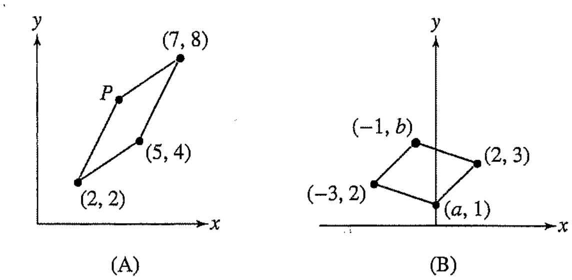

FIGURE 26

53. Let $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$, where A, B, C are three distinct points in the plane. Match (a)–(d) with (i)–(iv). (*Hint:* Draw a picture.)

(a) $-\mathbf{w}$ **(b)** $-\mathbf{v}$ **(c)** $\mathbf{w} - \mathbf{v}$ **(d)** $\mathbf{v} - \mathbf{w}$
(i) $\overrightarrow{CB}$ **(ii)** $\overrightarrow{CA}$ **(iii)** $\overrightarrow{BC}$ **(iv)** $\overrightarrow{BA}$

54. Find the components and length of the following vectors:

(a) $4\mathbf{i} + 3\mathbf{j}$ **(b)** $2\mathbf{i} - 3\mathbf{j}$ **(c)** $\mathbf{i} + \mathbf{j}$ **(d)** $\mathbf{i} - 3\mathbf{j}$

In Exercises 55–58, calculate the linear combination.

55. $3\mathbf{j} + (9\mathbf{i} + 4\mathbf{j})$

56. $-\frac{3}{2}\mathbf{i} + 5\left(\frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{i}\right)$

57. $(3\mathbf{i} + \mathbf{j}) - 6\mathbf{j} + 2(\mathbf{j} - 4\mathbf{i})$

58. $3(3\mathbf{i} - 4\mathbf{j}) + 5(\mathbf{i} + 4\mathbf{j})$

59. For each of the position vectors $\mathbf{u}$ with endpoints A, B, and C in Figure 27, indicate with a diagram the multiples $r\mathbf{v}$ and $s\mathbf{w}$ such that $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$. A sample is shown for $\mathbf{u} = \overrightarrow{OQ}$.

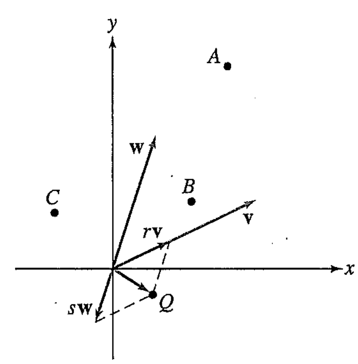

FIGURE 27

60. Sketch the parallelogram spanned by $\mathbf{v} = \langle 1, 4\rangle$ and $\mathbf{w} = \langle 5, 2\rangle$. Add the vector $\mathbf{u} = \langle 2, 3\rangle$ to the sketch and express $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

In Exercises 61 and 62, express $\mathbf{u}$ as a linear combination $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$. Then sketch $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and the parallelogram formed by $r\mathbf{v}$ and $s\mathbf{w}$.

61. $\mathbf{u} = \langle 3, -1\rangle$; $\mathbf{v} = \langle 2, 1\rangle$, $\mathbf{w} = \langle 1, 3\rangle$

62. $\mathbf{u} = \langle 6, -2\rangle$; $\mathbf{v} = \langle 1, 1\rangle$, $\mathbf{w} = \langle 1, -1\rangle$

12.2 SUMMARY

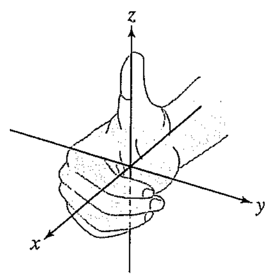

FIGURE 15

- The axes in $\mathbf{R}^3$ are labeled so that they satisfy the *right-hand rule*: When the fingers of your right hand curl from the positive x-axis toward the positive y-axis, your thumb points in the positive z-direction (Figure 15).
-

Sphere of radius R and center (a, b, c)	$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$
Cylinder of radius R with vertical axis through $(a, b, 0)$	$(x-a)^2 + (y-b)^2 = R^2$

- The notation and terminology for vectors in the plane carry over to vectors in $\mathbf{R}^3$.
- The length (or magnitude) of $\mathbf{v} = \overrightarrow{PQ}$, where $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$, is

$$\|\mathbf{v}\| = \|\overrightarrow{PQ}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$$

- Equations for the line through $P_0 = (x_0, y_0, z_0)$ with direction vector $\mathbf{v} = \langle a, b, c \rangle$:

Vector parametrization: $\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$

Parametric equations: $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$

- To obtain the line through $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$, take direction vector $\mathbf{v} = \overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$, and use the parametrization above. The segment $\overline{PQ}$ is parametrized by $\mathbf{r}(t)$ for $0 \le t \le 1$.

12.2 EXERCISES

Preliminary Questions

1. What is the terminal point of the vector $\mathbf{v} = \langle 3, 2, 1 \rangle$ based at the point $P = (1, 1, 1)$?

2. What are the components of the vector $\mathbf{v} = \langle 3, 2, 1 \rangle$ based at the point $P = (1, 1, 1)$?

3. If $\mathbf{v} = -3\mathbf{w}$, then (choose the correct answer):
(a) $\mathbf{v}$ and $\mathbf{w}$ are parallel.
(b) $\mathbf{v}$ and $\mathbf{w}$ point in the same direction.

4. Which of the following is a direction vector for the line through $P = (3, 2, 1)$ and $Q = (1, 1, 1)$?
(a) $\langle 3, 2, 1 \rangle$ (b) $\langle 1, 1, 1 \rangle$ (c) $\langle 2, 1, 0 \rangle$

5. How many different direction vectors does a line have?

6. True or false? If $\mathbf{v}$ is a direction vector for a line $\mathcal{L}$, then $-\mathbf{v}$ is also a direction vector for $\mathcal{L}$.

Exercises

1. Sketch the vector $\mathbf{v} = \langle 1, 3, 2 \rangle$ and compute its length.

2. Let $\mathbf{v} = \overrightarrow{P_0Q_0}$, where $P_0 = (1, -2, 5)$ and $Q_0 = (0, 1, -4)$. Which of the following vectors (with tail P and head Q) are equivalent to $\mathbf{v}$?

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
P	$(1, 2, 4)$	$(1, 5, 4)$	$(0, 0, 0)$	$(2, 4, 5)$
Q	$(0, 5, -5)$	$(0, -8, 13)$	$(-1, 3, -9)$	$(1, 7, 4)$

3. Sketch the vector $\mathbf{v} = \langle 1, 1, 0 \rangle$ based at $P = (0, 1, 1)$. Describe this vector in the form $\overrightarrow{PQ}$ for some point Q, and sketch the vector $\mathbf{v}_0$ based at the origin equivalent to $\mathbf{v}$.

4. Determine whether the coordinate systems (A)–(C) in Figure 16 satisfy the right-hand rule.

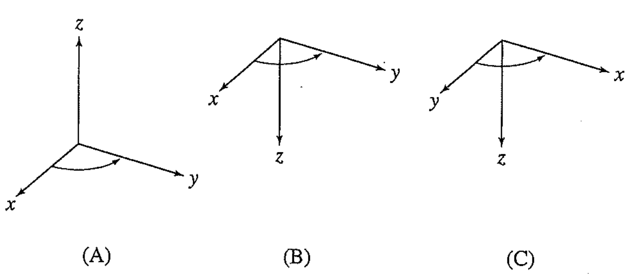

FIGURE 16

In Exercises 5–8, find the components of the vector $\overrightarrow{PQ}$.

5. $P = (1, 0, 1), \quad Q = (2, 1, 0)$

6. $P = (-3, -4, 2), \quad Q = (1, -4, 3)$

7. $P = (4, 6, 0), \quad Q = \left(-\frac{1}{2}, \frac{9}{2}, 1\right)$

8. $P = \left(-\frac{1}{2}, \frac{9}{2}, 1\right), \quad Q = (4, 6, 0)$

In Exercises 9–12, let $R = (1, 4, 3)$.

9. Calculate the length of $\overrightarrow{OR}$.

10. Find the point Q such that $\mathbf{v} = \overrightarrow{RQ}$ has components $\langle 4, 1, 1\rangle$, and sketch $\mathbf{v}$.

11. Find the point P such that $\mathbf{w} = \overrightarrow{PR}$ has components $\langle 3, -2, 3\rangle$, and sketch $\mathbf{w}$.

12. Find the components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2, 2)$.

13. Let $\mathbf{v} = \langle 4, 8, 12\rangle$. Which of the following vectors is parallel to $\mathbf{v}$? Which point in the same direction?

(a) $\langle 2, 4, 6\rangle$ **(b)** $\langle -1, -2, 3\rangle$
(c) $\langle -7, -14, -21\rangle$ **(d)** $\langle 6, 10, 14\rangle$

In Exercises 14–17, determine whether $\overrightarrow{AB}$ is equivalent to $\overrightarrow{PQ}$.

14. $A = (1, 1, 1)$ $B = (3, 3, 3)$
$P = (1, 4, 5)$ $Q = (3, 6, 7)$

15. $A = (1, 4, 1)$ $B = (-2, 2, 0)$
$P = (2, 5, 7)$ $Q = (-3, 2, 1)$

16. $A = (0, 0, 0)$ $B = (-4, 2, 3)$
$P = (4, -2, -3)$ $Q = (0, 0, 0)$

17. $A = (1, 1, 0)$ $B = (3, 3, 5)$
$P = (2, -9, 7)$ $Q = (4, -7, 13)$

In Exercises 18–23, calculate the linear combinations.

18. $5\langle 2, 2, -3\rangle + 3\langle 1, 7, 2\rangle$ **19.** $-2\langle 8, 11, 3\rangle + 4\langle 2, 1, 1\rangle$

20. $6(4\mathbf{j} + 2\mathbf{k}) - 3(2\mathbf{i} + 7\mathbf{k})$ **21.** $\frac{1}{2}\langle 4, -2, 8\rangle - \frac{1}{3}\langle 12, 3, 3\rangle$

22. $5(\mathbf{i} + 2\mathbf{j}) - 3(2\mathbf{j} + \mathbf{k}) + 7(2\mathbf{k} - \mathbf{i})$

23. $4\langle 6, -1, 1\rangle - 2\langle 1, 0, -1\rangle + 3\langle -2, 1, 1\rangle$

In Exercises 24–27, determine whether or not the two vectors are parallel.

24. $\mathbf{u} = \langle 1, -2, 5\rangle$, $\mathbf{v} = \langle -2, 4, -10\rangle$

25. $\mathbf{u} = \langle 4, 2, -6\rangle$, $\mathbf{v} = \langle 2, -1, 3\rangle$

26. $\mathbf{u} = \langle 4, 2, -6\rangle$, $\mathbf{v} = \langle 2, 1, 3\rangle$

27. $\mathbf{u} = \langle -3, 1, 4\rangle$, $\mathbf{v} = \langle 6, -2, 8\rangle$

In Exercises 28–31, find the given vector.

28. $\mathbf{e_v}$, where $\mathbf{v} = \langle 1, 1, 2\rangle$

29. $\mathbf{e_w}$, where $\mathbf{w} = \langle 4, -2, -1\rangle$

30. Unit vector in the direction of $\mathbf{u} = \langle 1, 0, 7\rangle$

31. Unit vector in the direction opposite to $\mathbf{v} = \langle -4, 4, 2\rangle$

32. Sketch the following vectors, and find their components and lengths:

(a) $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ **(b)** $\mathbf{i} + \mathbf{j} + \mathbf{k}$
(c) $4\mathbf{j} + 3\mathbf{k}$ **(d)** $12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

In Exercises 33–40, find a vector parametrization for the line with the given description.

33. Passes through $P = (1, 2, -8)$, direction vector $\mathbf{v} = \langle 2, 1, 3\rangle$

34. Passes through $P = (4, 0, 8)$, direction vector $\mathbf{v} = \langle 1, 0, 1\rangle$

35. Passes through $P = (4, 0, 8)$, direction vector $\mathbf{v} = 7\mathbf{i} + 4\mathbf{k}$

36. Passes through O, direction vector $\mathbf{v} = \langle 3, -1, -4\rangle$

37. Passes through $(1, 1, 1)$ and $(3, -5, 2)$

38. Passes through $(-2, 0, -2)$ and $(4, 3, 7)$

39. Passes through O and $(4, 1, 1)$

40. Passes through $(1, 1, 1)$ parallel to the line through $(2, 0, -1)$ and $(4, 1, 3)$

In Exercises 41–44, find parametric equations for the lines with the given description.

41. Perpendicular to the xy-plane, passes through the origin

42. Perpendicular to the yz-plane, passes through $(0, 0, 2)$

43. Parallel to the line through $(1, 1, 0)$ and $(0, -1, -2)$, passes through $(0, 0, 4)$

44. Passes through $(1, -1, 0)$ and $(0, -1, 2)$

45. Which of the following is a parametrization of the line through $P = (4, 9, 8)$ perpendicular to the xz-plane (Figure 17)?

(a) $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 1, 0, 1\rangle$ **(b)** $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 0, 0, 1\rangle$
(c) $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 0, 1, 0\rangle$ **(d)** $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 1, 1, 0\rangle$

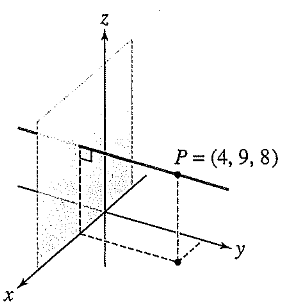

FIGURE 17

46. Find a parametrization of the line through $P = (4, 9, 8)$ perpendicular to the yz-plane.

47. Show that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ define the same line, where

$$\mathbf{r}_1(t) = \langle 3, -1, 4\rangle + t\langle 8, 12, -6\rangle$$
$$\mathbf{r}_2(t) = \langle 11, 11, -2\rangle + t\langle 4, 6, -3\rangle$$

Hint: Show that $\mathbf{r}_2(t)$ passes through $(3, -1, 4)$ and that the direction vectors for $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are parallel.

48. Show that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ define the same line, where

$$\mathbf{r}_1(t) = t\langle 2, 1, 3\rangle, \qquad \mathbf{r}_2(t) = \langle -6, -3, -9\rangle + t\langle 8, 4, 12\rangle$$

49. Find two different vector parametrizations of the line through $P = (5, 5, 2)$ with direction vector $\mathbf{v} = \langle 0, -2, 1\rangle$.

50. Find the point of intersection of the lines $\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\langle -3, 1, 0\rangle$ and $\mathbf{s}(t) = \langle 0, 1, 1\rangle + t\langle 2, 0, 1\rangle$.

51. Show that the lines $\mathbf{r}_1(t) = \langle -1, 2, 2\rangle + t\langle 4, -2, 1\rangle$ and $\mathbf{r}_2(t) = \langle 0, 1, 1\rangle + t\langle 2, 0, 1\rangle$ do not intersect.

52. Determine whether the lines $\mathbf{r}_1(t) = \langle 2, 1, 1\rangle + t\langle -4, 0, 1\rangle$ and $\mathbf{r}_2(s) = \langle -4, 1, 5\rangle + s\langle 2, 1, -2\rangle$ intersect, and if so, find the point of intersection.

53. Determine whether the lines $\mathbf{r}_1(t) = \langle 0, 1, 1\rangle + t\,\langle 1, 1, 2\rangle$ and $\mathbf{r}_2(s) = \langle 2, 0, 3\rangle + s\,\langle 1, 4, 4\rangle$ intersect, and if so, find the point of intersection.

54. Find the intersection of the lines $\mathbf{r}_1(t) = \langle -1, 1\rangle + t\,\langle 2, 4\rangle$ and $\mathbf{r}_2(s) = \langle 2, 1\rangle + s\,\langle -1, 6\rangle$ in $\mathbf{R}^2$.

55. A meteor follows a trajectory $\mathbf{r}(t) = \langle 2, 1, 4\rangle + t\,\langle 3, 2, -1\rangle$ km, with t in minutes, near the surface of the earth, which is represented by the xy-plane. Determine at what time the meteor hits the ground.

56. A laser's beam shines along the ray given by $\mathbf{r}_1(t) = \langle 1, 2, 4\rangle + t\langle 2, 1, -1\rangle$ for $t \ge 0$. A second laser's beam shines along the ray given by $\mathbf{r}_2(s) = \langle 6, 3, -1\rangle + s\langle -5, 2, c\rangle$ for $s \ge 0$, where the value of c allows for the adjustment of the z-coordinate of its direction vector. Find the value of c that will make the two beams intersect.

57. Find the components of the vector $\mathbf{v}$ whose tail and head are the midpoints of segments $\overline{AC}$ and $\overline{BC}$ in Figure 18. [Note that the midpoint of (a_1, a_2, a_3) and (b_1, b_2, b_3) is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$.]

58. Find the components of the vector $\mathbf{w}$ whose tail is C and head is the midpoint of $\overline{AB}$ in Figure 18.

59. A box that weighs 1000 kg is hanging from a crane at the dock. The crane has a square 20 m by 20 m framework as in Figure 19, with four cables, each of the same length, supporting the box. The box hangs 10 m below the level of the framework. Find the magnitude of the force acting on each cable.

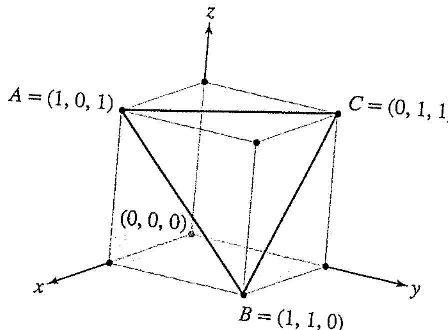

FIGURE 18

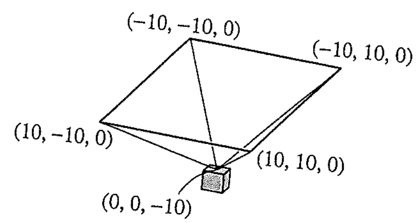

FIGURE 19

Further Insights and Challenges

*In Exercises 60–66, we consider the equations of a line in **symmetric form**, when $a \ne 0$, $b \ne 0$, $c \ne 0$.*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad \boxed{10}$$

60. Let $\mathcal{L}$ be the line through $P_0 = (x_0, y_0, z_0)$ with direction vector $\mathbf{v} = \langle a, b, c\rangle$. Show that $\mathcal{L}$ is defined by the symmetric equations (10). *Hint:* Use the vector parametrization to show that every point on $\mathcal{L}$ satisfies (10).

61. Find the symmetric equations of the line through $P_0 = (-2, 3, 3)$ with direction vector $\mathbf{v} = \langle 2, 4, 3\rangle$.

62. Find the symmetric equations of the line through $P = (1, 1, 2)$ and $Q = (-2, 4, 0)$.

63. Find the symmetric equations of the line

$$x = 3 + 2t, \quad y = 4 - 9t, \quad z = 12t$$

64. Find a vector parametrization for the line

$$\frac{x - 5}{9} = \frac{y + 3}{7} = z - 10$$

65. Find a vector parametrization for the line $\frac{x}{2} = \frac{y}{7} = \frac{z}{8}$.

66. Show that the line in the plane through (x_0, y_0) of slope m has symmetric equations

$$x - x_0 = \frac{y - y_0}{m}$$

67. A median of a triangle is a segment joining a vertex to the midpoint of the opposite side. Referring to Figure 20(A), prove that three medians of triangle ABC intersect at the terminal point P of the vector $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. The point P is the *centroid* of the triangle. *Hint:* Show, by parametrizing the segment $\overline{AA'}$, that P lies two-thirds of the way from A to A'. It will follow similarly that P lies on the other two medians.

68. A median of a tetrahedron is a segment joining a vertex to the centroid of the opposite face. The tetrahedron in Figure 20(B) has vertices at the origin and at the terminal points of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Show that the medians intersect at the terminal point of $\frac{1}{4}(\mathbf{u} + \mathbf{v} + \mathbf{w})$.

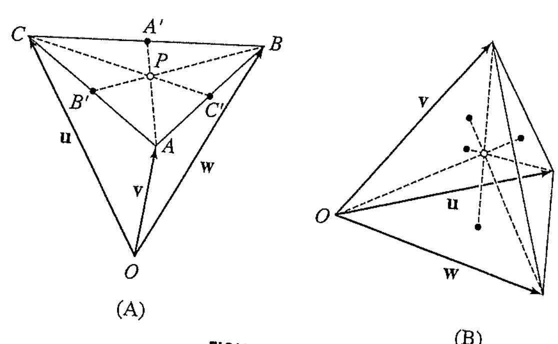

FIGURE 20

12.3 EXERCISES

Preliminary Questions

1. Is the dot product of two vectors a scalar or a vector?

2. What can you say about the angle between $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} < 0$?

3. Which property of dot products allows us to conclude that if $\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{w}$, then $\mathbf{v}$ is orthogonal to $\mathbf{u} + \mathbf{w}$?

4. Which is the projection of $\mathbf{v}$ along $\mathbf{v}$: (a) $\mathbf{v}$ or (b) $\mathbf{e_v}$?

5. Let $\mathbf{u}_{||\mathbf{v}}$ be the projection of $\mathbf{u}$ along $\mathbf{v}$. Which of the following is the projection $\mathbf{u}$ along the vector $2\mathbf{v}$ and which is the projection of $2\mathbf{u}$ along $\mathbf{v}$?
(a) $\frac{1}{2}\mathbf{u}_{||\mathbf{v}}$ (b) $\mathbf{u}_{||\mathbf{v}}$ (c) $2\mathbf{u}_{||\mathbf{v}}$

6. Which of the following is equal to $\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$?
(a) $\mathbf{u} \cdot \mathbf{v}$ (b) $\mathbf{u} \cdot \mathbf{e_v}$ (c) $\mathbf{e_u} \cdot \mathbf{e_v}$

Exercises

In Exercises 1–12, compute the dot product.

1. $\langle 1, 2, 1\rangle \cdot \langle 4, 3, 5\rangle$
2. $\langle 3, -2, 2\rangle \cdot \langle 1, 0, 1\rangle$
3. $\langle 0, 1, 0\rangle \cdot \langle 7, 41, -3\rangle$
4. $\langle 1, 1, 1\rangle \cdot \langle 6, 4, 2\rangle$
5. $\langle 3, 1\rangle \cdot \langle 4, -7\rangle$
6. $\langle \frac{1}{6}, \frac{1}{2}\rangle \cdot \langle 3, \frac{1}{2}\rangle$
7. $\mathbf{k} \cdot \mathbf{j}$
8. $\mathbf{k} \cdot \mathbf{k}$
9. $(\mathbf{i}+\mathbf{j}) \cdot (\mathbf{j}+\mathbf{k})$
10. $(3\mathbf{j}+2\mathbf{k}) \cdot (\mathbf{i}-4\mathbf{k})$
11. $(\mathbf{i}+\mathbf{j}+\mathbf{k}) \cdot (3\mathbf{i}+2\mathbf{j}-5\mathbf{k})$
12. $(-\mathbf{k}) \cdot (\mathbf{i}-2\mathbf{j}+7\mathbf{k})$

In Exercises 13–18, determine whether the two vectors are orthogonal and, if not, whether the angle between them is acute or obtuse.

13. $\langle 1, 1, 1\rangle$, $\langle 1, -2, -2\rangle$
14. $\langle 0, 2, 4\rangle$, $\langle -5, 0, 0\rangle$
15. $\langle 1, 2, 1\rangle$, $\langle 7, -3, -1\rangle$
16. $\langle 0, 2, 4\rangle$, $\langle 3, 1, 0\rangle$
17. $\langle \frac{12}{5}, -\frac{4}{5}\rangle$, $\langle \frac{1}{2}, -\frac{7}{4}\rangle$
18. $\langle 12, 6\rangle$, $\langle 2, -4\rangle$

In Exercises 19–22, find the cosine of the angle between the vectors.

19. $\langle 0, 3, 1\rangle$, $\langle 4, 0, 0\rangle$
20. $\langle 1, 1, 1\rangle$, $\langle 2, -1, 2\rangle$
21. $\mathbf{i}+\mathbf{j}$, $\mathbf{j}+2\mathbf{k}$
22. $3\mathbf{i}+\mathbf{k}$, $\mathbf{i}+\mathbf{j}+\mathbf{k}$

In Exercises 23–28, find the angle between the vectors. Use a calculator if necessary.

23. $\langle 2, \sqrt{2}\rangle$, $\langle 1+\sqrt{2}, 1-\sqrt{2}\rangle$
24. $\langle 5, \sqrt{3}\rangle$, $\langle \sqrt{3}, 2\rangle$
25. $\langle 1, 1, 1\rangle$, $\langle 1, 0, 1\rangle$
26. $\langle 3, 1, 1\rangle$, $\langle 2, -4, 2\rangle$
27. $\langle 0, 1, 1\rangle$, $\langle 1, -1, 0\rangle$
28. $\langle 1, 1, -1\rangle$, $\langle 1, -2, -1\rangle$

29. Find all values of b for which the vectors are orthogonal.
(a) $\langle b, 3, 2\rangle$, $\langle 1, b, 1\rangle$
(b) $\langle 4, -2, 7\rangle$, $\langle b^2, b, 0\rangle$

30. Find a vector that is orthogonal to $\langle -1, 2, 2\rangle$.

31. Find two vectors that are not multiples of each other and are both orthogonal to $\langle 2, 0, -3\rangle$.

32. Find a vector that is orthogonal to $\mathbf{v} = \langle 1, 2, 1\rangle$ but not to $\mathbf{w} = \langle 1, 0, -1\rangle$.

33. Find $\mathbf{v} \cdot \mathbf{e}$ where $\|\mathbf{v}\| = 3$, $\mathbf{e}$ is a unit vector, and the angle between $\mathbf{e}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

34. Assume that $\mathbf{v}$ lies in the yz-plane. Which of the following dot products is equal to zero for all choices of $\mathbf{v}$?
(a) $\mathbf{v} \cdot \langle 0, 2, 1\rangle$
(b) $\mathbf{v} \cdot \mathbf{k}$
(c) $\mathbf{v} \cdot \langle -3, 0, 0\rangle$
(d) $\mathbf{v} \cdot \mathbf{j}$

In Exercises 35–38, simplify the expression.

35. $(\mathbf{v}-\mathbf{w}) \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w}$
36. $(\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w}) - 2\mathbf{v} \cdot \mathbf{w}$
37. $(\mathbf{v}+\mathbf{w}) \cdot \mathbf{v} - (\mathbf{v}+\mathbf{w}) \cdot \mathbf{w}$
38. $(\mathbf{v}+\mathbf{w}) \cdot \mathbf{v} - (\mathbf{v}-\mathbf{w}) \cdot \mathbf{w}$

In Exercises 39–42, use the properties of the dot product to evaluate the expression, assuming that $\mathbf{u} \cdot \mathbf{v} = 2$, $\|\mathbf{u}\| = 1$, and $\|\mathbf{v}\| = 3$.

39. $\mathbf{u} \cdot (4\mathbf{v})$
40. $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{v}$
41. $2\mathbf{u} \cdot (3\mathbf{u}-\mathbf{v})$
42. $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v})$

43. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\,\|\mathbf{w}\|$.

44. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\|\mathbf{v}\|\,\|\mathbf{w}\|$.

45. Assume that $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 5$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\theta = \frac{\pi}{3}$.
(a) Use the relation $\|\mathbf{v}+\mathbf{w}\|^2 = (\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w})$ to show that $\|\mathbf{v}+\mathbf{w}\|^2 = 3^2 + 5^2 + 2\mathbf{v} \cdot \mathbf{w}$.
(b) Find $\|\mathbf{v}+\mathbf{w}\|$.

46. Assume that $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $120°$. Determine:
(a) $\mathbf{v} \cdot \mathbf{w}$ (b) $\|2\mathbf{v}+\mathbf{w}\|$ (c) $\|2\mathbf{v}-3\mathbf{w}\|$

47. Show that if $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e}+\mathbf{f}\| = \frac{3}{2}$, then $\|\mathbf{e}-\mathbf{f}\| = \frac{\sqrt{7}}{2}$. *Hint:* Show that $\mathbf{e} \cdot \mathbf{f} = \frac{1}{8}$.

48. Find $\|2\mathbf{e}-3\mathbf{f}\|$, assuming that $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e}+\mathbf{f}\| = \sqrt{3/2}$.

49. Find the angle θ in the triangle in Figure 12.

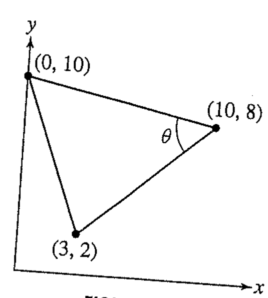

FIGURE 12

50. Find all three angles in the triangle in Figure 13.

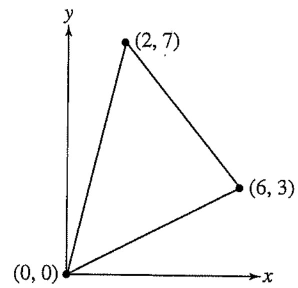

FIGURE 13

51. (a) Draw $\mathbf{u}_{\|\mathbf{v}}$ and $\mathbf{v}_{\|\mathbf{u}}$ for the vectors appearing as in Figure 14.
(b) Which of $\mathbf{u}_{\|\mathbf{v}}$ and $\mathbf{v}_{\|\mathbf{u}}$ has the greater magnitude?

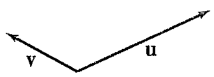

FIGURE 14

52. Let **u** and **v** be two nonzero vectors.
(a) Is it possible for the component of **u** along **v** to have the opposite sign from the component of **v** along **u**? Why or why not?
(b) What must be true of the vectors if either of these two components is 0?

In Exercises 53–60, find the projection of **u** *along* **v**.

53. $\mathbf{u} = \langle 2, 5\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

54. $\mathbf{u} = \langle 2, -3\rangle$, $\mathbf{v} = \langle 1, 2\rangle$

55. $\mathbf{u} = \langle -1, 2, 0\rangle$, $\mathbf{v} = \langle 2, 0, 1\rangle$

56. $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle 1, 1, 0\rangle$

57. $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = \mathbf{k}$

58. $\mathbf{u} = \mathbf{i} + 29\mathbf{k}$, $\mathbf{v} = \mathbf{j}$

59. $\mathbf{u} = \langle a, b, c\rangle$, $\mathbf{v} = \mathbf{i}$

60. $\mathbf{u} = \langle a, a, b\rangle$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

In Exercises 61 and 62, compute the component of **u** *along* **v**.

61. $\mathbf{u} = \langle 3, 2, 1\rangle$, $\mathbf{v} = \langle 1, 0, 1\rangle$

62. $\mathbf{u} = \langle 3, 0, 9\rangle$, $\mathbf{v} = \langle 1, 2, 2\rangle$

63. Find the length of $\overline{OP}$ in Figure 15.

64. Find $\|\mathbf{u}_{\perp\mathbf{v}}\|$ in Figure 15.

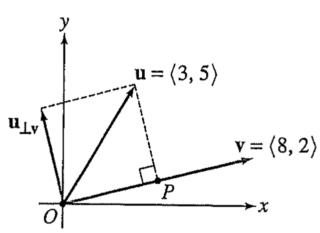

FIGURE 15

In Exercises 65–70, find the decomposition $\mathbf{a} = \mathbf{a}_{\|\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}}$ *with respect to* **b**.

65. $\mathbf{a} = \langle 1, 0\rangle$, $\mathbf{b} = \langle 1, 1\rangle$

66. $\mathbf{a} = \langle 2, -3\rangle$, $\mathbf{b} = \langle 5, 0\rangle$

67. $\mathbf{a} = \langle 4, -1, 0\rangle$, $\mathbf{b} = \langle 0, 1, 1\rangle$

68. $\mathbf{a} = \langle 4, -1, 5\rangle$, $\mathbf{b} = \langle 2, 1, 1\rangle$

69. $\mathbf{a} = \langle x, y\rangle$, $\mathbf{b} = \langle 1, -1\rangle$

70. $\mathbf{a} = \langle x, y, z\rangle$, $\mathbf{b} = \langle 1, 1, 1\rangle$

71. Let $\mathbf{e}_\theta = \langle \cos\theta, \sin\theta\rangle$. Show that $\mathbf{e}_\theta \cdot \mathbf{e}_\psi = \cos(\theta - \psi)$ for any two angles θ and ψ.

72. Let **v** and **w** be vectors in the plane.
(a) Use Theorem 2 to explain why the dot product $\mathbf{v} \cdot \mathbf{w}$ does not change if both **v** and **w** are rotated by the same angle θ.
(b) Sketch the vectors $\mathbf{e}_1 = \langle 1, 0\rangle$ and $\mathbf{e}_2 = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$, and determine the vectors $\mathbf{e}_1', \mathbf{e}_2'$ obtained by rotating $\mathbf{e}_1, \mathbf{e}_2$ through an angle $\frac{\pi}{4}$. Verify that $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1' \cdot \mathbf{e}_2'$.

In Exercises 73–76, refer to Figure 16.

73. Find the angle between $\overline{AB}$ and $\overline{AC}$.

74. Find the angle between $\overline{AB}$ and $\overline{AD}$.

75. Calculate the projection of $\overrightarrow{AC}$ along $\overrightarrow{AD}$.

76. Calculate the projection of $\overrightarrow{AD}$ along $\overrightarrow{AB}$.

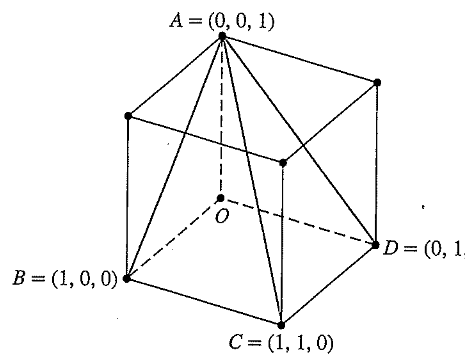

FIGURE 16 Unit cube in $\mathbf{R}^3$.

77. The methane molecule CH_4 consists of a carbon molecule bonded to four hydrogen molecules that are spaced as far apart from each other as possible. The hydrogen atoms then sit at the vertices of a tetrahedron, with the carbon atom at its center, as in Figure 17. We can model this with the carbon atom at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the hydrogen atoms at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$. Use the dot product to find the bond angle α formed between any two of the line segments from the carbon atom to the hydrogen atoms.

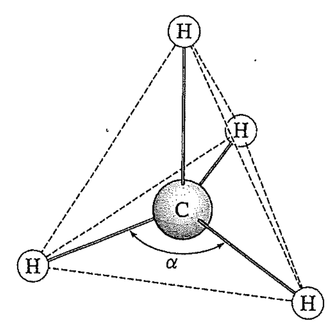

FIGURE 17 A methane molecule.

78. Iron forms a crystal lattice where each central atom appears at the center of a cube, the corners of which correspond to additional iron atoms, as in Figure 18. Use the dot product to find the angle β between the line segments from the central atom to two adjacent outer atoms. *Hint: Take the central atom to be situated at the origin and the corner atoms to occur at* $(\pm 1, \pm 1, \pm 1)$.

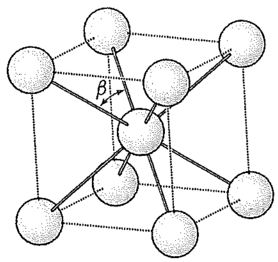

FIGURE 18 An iron crystal.

79. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and set $\mathbf{u} = \mathbf{e_v} + \mathbf{e_w}$. Use the dot product to show that the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $\mathbf{u}$ and $\mathbf{w}$. Explain this result geometrically with a diagram.

80. Let $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{a}$ be nonzero vectors such that $\mathbf{v} \cdot \mathbf{a} = \mathbf{w} \cdot \mathbf{a}$. Is it true that $\mathbf{v} = \mathbf{w}$? Either prove this or give a counterexample.

81. Calculate the force (in newtons) required to push a 40-kg wagon up a 10° incline (Figure 19).

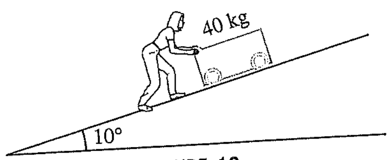

FIGURE 19

82. A force $\mathbf{F}$ is applied to each of two ropes (of negligible weight) attached to opposite ends of a 40-kg wagon and making an angle of 35° with the horizontal (Figure 20). What is the maximum magnitude of $\mathbf{F}$ (in newtons) that can be applied without lifting the wagon off the ground?

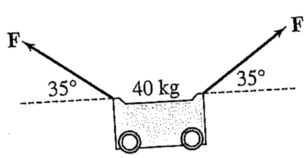

FIGURE 20

83. A light beam travels along the ray determined by a unit vector $\mathbf{L}$, strikes a flat surface at point P, and is reflected along the ray determined by a unit vector $\mathbf{R}$, where $\theta_1 = \theta_2$ (Figure 21). Show that if $\mathbf{N}$ is the unit vector orthogonal to the surface, then

$$\mathbf{R} = 2(\mathbf{L} \cdot \mathbf{N})\mathbf{N} - \mathbf{L}$$

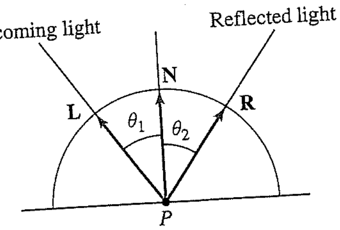

FIGURE 21

84. Let P and Q be antipodal (opposite) points on a sphere of radius r centered at the origin and let R be a third point on the sphere (Figure 22). Prove that $\overline{PR}$ and $\overline{QR}$ are orthogonal.

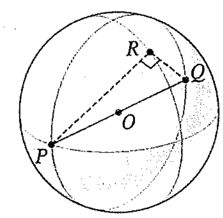

FIGURE 22

85. Prove that $\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = 4\mathbf{v} \cdot \mathbf{w}$.

86. Use Exercise 85 to show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\|\mathbf{v} - \mathbf{w}\| = \|\mathbf{v} + \mathbf{w}\|$.

87. Show that the two diagonals of a parallelogram are perpendicular if and only if its sides have equal length. *Hint:* Use Exercise 86 to show that $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are orthogonal if and only if $\|\mathbf{v}\| = \|\mathbf{w}\|$.

88. Verify the Distributive Law:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

89. Verify that $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w})$ for any scalar λ.

Further Insights and Challenges

90. Prove the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$, by referring to Figure 23. *Hint:* Consider the right triangle $\triangle PQR$.

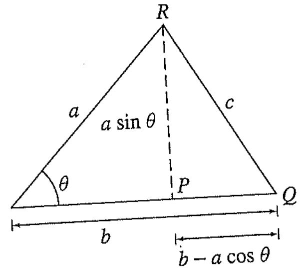

FIGURE 23

91. In this exercise, we prove the Cauchy–Schwarz inequality: If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then

$$|\mathbf{v} \cdot \mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$$

(a) Let $f(x) = \|x\mathbf{v} + \mathbf{w}\|^2$ for x a scalar. Show that $f(x) = ax^2 + bx + c$, where $a = \|\mathbf{v}\|^2$, $b = 2\mathbf{v} \cdot \mathbf{w}$, and $c = \|\mathbf{w}\|^2$.

(b) Conclude that $b^2 - 4ac \le 0$. *Hint:* Observe that $f(x) \ge 0$ for all x.

92. Use (6) to prove the Triangle Inequality:

$$\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$$

Hint: First use the Triangle Inequality for numbers to prove

$$|(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})| \le |(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v}| + |(\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}|$$

93. This exercise gives another proof of the relation between th dot product and the angle θ between two vectors $\mathbf{v} = \langle a_1, b_1 \rangle$ an $\mathbf{w} = \langle a_2, b_2 \rangle$ in the plane. Observe that $\mathbf{v} = \|\mathbf{v}\| \langle \cos\theta_1, \sin\theta_1 \rangle$ ar $\mathbf{w} = \|\mathbf{w}\| \langle \cos\theta_2, \sin\theta_2 \rangle$, with θ_1 and θ_2 as in Figure 24. Then use tl addition formula for the cosine to show that

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\| \cos\theta$$

- The cross product $\mathbf{v} \times \mathbf{w}$ is the unique vector with the following three properties:
  - **(i)** $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.
  - **(ii)** $\mathbf{v} \times \mathbf{w}$ has length $\|\mathbf{v}\|\,\|\mathbf{w}\| \sin\theta$ (θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, $0 \le \theta \le \pi$).
  - **(iii)** $\{\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}\}$ is a right-handed system.
- Properties of the cross product:
  - **(i)** $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
  - **(ii)** $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} = \lambda\mathbf{v}$ for some scalar or $\mathbf{v} = \mathbf{0}$
  - **(iii)** $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$
  - **(iv)** $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

    $\mathbf{v} \times (\mathbf{u} + \mathbf{w}) = \mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{w}$
- Cross products of standard basis vectors (Figure 15):

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

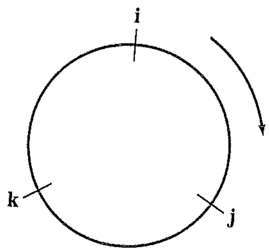

FIGURE 15 Circle for computing the cross products of the basis vectors.

- The parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ has area $\|\mathbf{v} \times \mathbf{w}\|$.
- The triangle spanned by $\mathbf{v}$ and $\mathbf{w}$ has area $\dfrac{\|\mathbf{v} \times \mathbf{w}\|}{2}$.
- Cross-product identity: $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2$.
- The *vector triple product* is defined by $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. We have

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}$$

- The parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ has volume $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

12.4 EXERCISES

Preliminary Questions

1. What is the (1, 3) minor of the matrix $\begin{vmatrix} 3 & 4 & 2 \\ -5 & -1 & 1 \\ 4 & 0 & 3 \end{vmatrix}$?

2. The angle between two unit vectors $\mathbf{e}$ and $\mathbf{f}$ is $\frac{\pi}{6}$. What is the length of $\mathbf{e} \times \mathbf{f}$?

3. What is $\mathbf{u} \times \mathbf{w}$, assuming that $\mathbf{w} \times \mathbf{u} = \langle 2, 2, 1 \rangle$?

4. Find the cross product without using the formula:
(a) $\langle 4, 8, 2 \rangle \times \langle 4, 8, 2 \rangle$ **(b)** $\langle 4, 8, 2 \rangle \times \langle 2, 4, 1 \rangle$

5. What are $\mathbf{i} \times \mathbf{j}$ and $\mathbf{i} \times \mathbf{k}$?

6. When is the cross product $\mathbf{v} \times \mathbf{w}$ equal to zero?

7. Which of the following are meaningful and which are not? Explain.
(a) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$
(b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
(c) $\|\mathbf{w}\|(\mathbf{u} \cdot \mathbf{v})$
(d) $\|\mathbf{w}\|(\mathbf{u} \times \mathbf{v})$

8. Which of the following vectors are equal to $\mathbf{j} \times \mathbf{i}$?
(a) $\mathbf{i} \times \mathbf{k}$
(b) $-\mathbf{k}$
(c) $\mathbf{i} \times \mathbf{j}$

Exercises

In Exercises 1–4, calculate the 2×2 determinant.

1. $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$ **2.** $\begin{vmatrix} \frac{2}{3} & \frac{1}{6} \\ -5 & 2 \end{vmatrix}$

3. $\begin{vmatrix} -6 & 9 \\ 1 & 1 \end{vmatrix}$ **4.** $\begin{vmatrix} 9 & 25 \\ 5 & 14 \end{vmatrix}$

In Exercises 5–8, calculate the 3×3 determinant.

5. $\begin{vmatrix} 1 & 2 & 1 \\ 4 & -3 & 0 \\ 1 & 0 & 1 \end{vmatrix}$ **6.** $\begin{vmatrix} 1 & 0 & 1 \\ -2 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -4 & 2 \end{vmatrix}$ **8.** $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$

In Exercises 9–12, calculate $\mathbf{v} \times \mathbf{w}$.

9. $\mathbf{v} = \langle 1, 2, 1 \rangle$, $\mathbf{w} = \langle 3, 1, 1 \rangle$

10. $\mathbf{v} = \langle 2, 0, 0 \rangle$, $\mathbf{w} = \langle -1, 0, 1 \rangle$

11. $\mathbf{v} = \langle \frac{2}{3}, 1, \frac{1}{2} \rangle$, $\mathbf{w} = \langle 4, -6, 3 \rangle$

12. $\mathbf{v} = \langle 1, 1, 0 \rangle$, $\mathbf{w} = \langle 0, 1, 1 \rangle$

In Exercises 13–16, use the relations in Eq. (5) to calculate the cross product.

13. $(\mathbf{i}+\mathbf{j})\times\mathbf{k}$

14. $(\mathbf{j}-\mathbf{k})\times(\mathbf{j}+\mathbf{k})$

15. $(\mathbf{i}-3\mathbf{j}+2\mathbf{k})\times(\mathbf{j}-\mathbf{k})$

16. $(2\mathbf{i}-3\mathbf{j}+4\mathbf{k})\times(\mathbf{i}+\mathbf{j}-7\mathbf{k})$

In Exercises 17–22, calculate the cross product assuming that

$$\mathbf{u}\times\mathbf{v}=\langle 1,1,0\rangle,\quad \mathbf{u}\times\mathbf{w}=\langle 0,3,1\rangle,\quad \mathbf{v}\times\mathbf{w}=\langle 2,-1,1\rangle$$

17. $\mathbf{v}\times\mathbf{u}$

18. $\mathbf{v}\times(\mathbf{u}+\mathbf{v})$

19. $\mathbf{w}\times(\mathbf{u}+\mathbf{v})$

20. $(3\mathbf{u}+4\mathbf{w})\times\mathbf{w}$

21. $(\mathbf{u}-2\mathbf{v})\times(\mathbf{u}+2\mathbf{v})$

22. $(\mathbf{v}+\mathbf{w})\times(3\mathbf{u}+2\mathbf{v})$

23. Let $\mathbf{v}=\langle a,b,c\rangle$. Calculate $\mathbf{v}\times\mathbf{i}$, $\mathbf{v}\times\mathbf{j}$, and $\mathbf{v}\times\mathbf{k}$.

24. Find $\mathbf{v}\times\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are vectors of length 3 in the xz-plane, oriented as in Figure 16, and $\theta=\frac{\pi}{6}$.

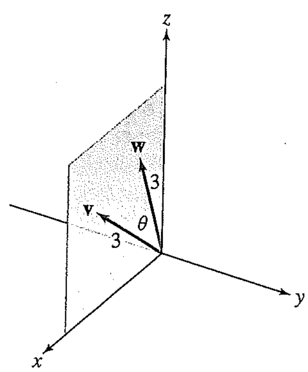

FIGURE 16

In Exercises 25 and 26, refer to Figure 17.

25. Which of $\mathbf{u}$ and $-\mathbf{u}$ is equal to $\mathbf{v}\times\mathbf{w}$?

26. Which of the following form a right-handed system?

(a) $\{\mathbf{v},\mathbf{w},\mathbf{u}\}$ (b) $\{\mathbf{w},\mathbf{v},\mathbf{u}\}$ (c) $\{\mathbf{v},\mathbf{u},\mathbf{w}\}$

(d) $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ (e) $\{\mathbf{w},\mathbf{v},-\mathbf{u}\}$ (f) $\{\mathbf{v},-\mathbf{u},\mathbf{w}\}$

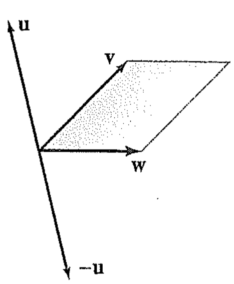

FIGURE 17

27. Let $\mathbf{v}=\langle 3,0,0\rangle$ and $\mathbf{w}=\langle 0,1,-1\rangle$. Determine $\mathbf{u}=\mathbf{v}\times\mathbf{w}$ using the geometric properties of the cross product rather than the formula.

28. What are the possible angles θ between two unit vectors $\mathbf{e}$ and $\mathbf{f}$ if $\|\mathbf{e}\times\mathbf{f}\|=\frac{1}{2}$?

29. Show that if $\mathbf{v}$ and $\mathbf{w}$ lie in the yz-plane, then $\mathbf{v}\times\mathbf{w}$ is a multiple of $\mathbf{i}$.

30. Find the two unit vectors orthogonal to both $\mathbf{a}=\langle 3,1,1\rangle$ and $\mathbf{b}=\langle -1,2,1\rangle$.

31. Let $\mathbf{e}$ and $\mathbf{e}'$ be unit vectors in $\mathbf{R}^3$ such that $\mathbf{e}\perp\mathbf{e}'$. Use the geometric properties of the cross product to compute $\mathbf{e}\times(\mathbf{e}'\times\mathbf{e})$.

32. Calculate the force $\mathbf{F}$ on an electron (charge $q=-1.6\times 10^{-19}$ C) moving with velocity 10^5 m/s in the direction $\mathbf{i}$ in a uniform magnetic field $\mathbf{B}$, where $\mathbf{B}=0.0004\mathbf{i}+0.0001\mathbf{j}$ teslas (see Example 5).

33. An electron moving with velocity $\mathbf{v}$ in the plane experiences a force $\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$, where q is the charge on the electron and $\mathbf{B}$ is a uniform magnetic field pointing directly out of the page. Which of the two vectors $\mathbf{F}_1$ or $\mathbf{F}_2$ in Figure 18 represents the force on the electron? Remember that q is negative.

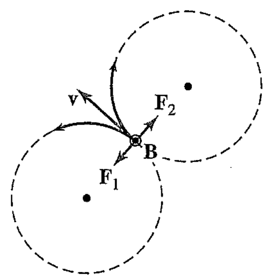

FIGURE 18 The magnetic field vector $\mathbf{B}$ points directly out of the page.

34. Calculate the scalar triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, where $\mathbf{u}=\langle 1,1,0\rangle$, $\mathbf{v}=\langle 3,-2,2\rangle$, and $\mathbf{w}=\langle 4,-1,2\rangle$.

35. Verify identity (12) for vectors $\mathbf{v}=\langle 3,-2,2\rangle$ and $\mathbf{w}=\langle 4,-1,2\rangle$.

36. Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in Figure 19.

37. Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ in Figure 19.

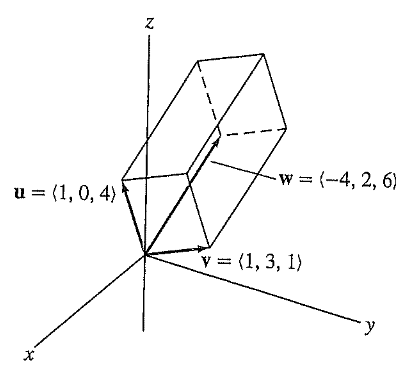

FIGURE 19

38. Calculate the volume of the parallelepiped spanned by

$$\mathbf{u}=\langle 2,2,1\rangle,\quad \mathbf{v}=\langle 1,0,3\rangle,\quad \mathbf{w}=\langle 0,-4,0\rangle$$

39. Sketch and compute the volume of the parallelepiped spanned by

$$\mathbf{u}=\langle 1,0,0\rangle,\quad \mathbf{v}=\langle 0,2,0\rangle,\quad \mathbf{w}=\langle 1,1,2\rangle$$

40. Sketch the parallelogram spanned by $\mathbf{u}=\langle 1,1,1\rangle$ and $\mathbf{v}=\langle 0,0,4\rangle$, and compute its area.

41. Calculate the area of the parallelogram spanned by $\mathbf{u}=\langle 1,0,3\rangle$ and $\mathbf{v}=\langle 2,1,1\rangle$.

42. Find the area of the parallelogram determined by the vectors $\langle a, 0, 0\rangle$ and $\langle 0, b, c\rangle$.

43. Sketch the triangle with vertices at the origin O, $P = (3, 3, 0)$, and $Q = (0, 3, 3)$, and compute its area using cross products.

44. Use the cross product to find the area of the triangle with vertices $P = (1, 1, 5)$, $Q = (3, 4, 3)$, and $R = (1, 5, 7)$ (Figure 20).

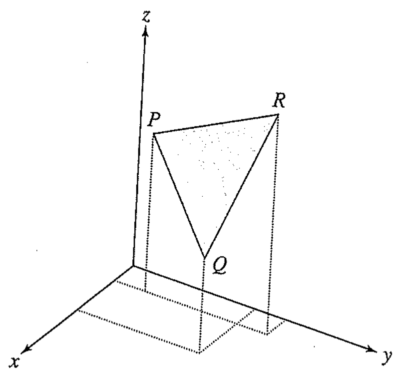

FIGURE 20

45. Use cross products to find the area of the triangle in the xy-plane defined by $(1, 2)$, $(3, 4)$, and $(-2, 2)$.

46. Use cross products to find the area of the quadrilateral in the xy-plane defined by $(0, 0)$, $(1, -1)$, $(3, 1)$, and $(2, 4)$.

47. Check that the four points $P(2, 4, 4)$, $Q(3, 1, 6)$, $R(2, 8, 0)$, and $S(7, 2, 1)$ all lie in a plane. Then use vectors to find the area of the quadrilateral they define.

48. Find three nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \neq \mathbf{0}$ but $\mathbf{b} \neq \mathbf{c}$.

In Exercises 49–51, verify the identity using the formula for the cross product.

49. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

50. $(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w})$ (λ a scalar)

51. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

52. Use the geometric description in Theorem 1 to prove Theorem 2 (iii): $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} = \lambda \mathbf{v}$ for some scalar λ or $\mathbf{v} = \mathbf{0}$.

53. Verify the relations (5).

54. Show that

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{j} \times \mathbf{j})$$

Conclude that the Associative Law does not hold for cross products.

55. The components of the cross product have a geometric interpretation. Show that the absolute value of the $\mathbf{k}$-component of $\mathbf{v} \times \mathbf{w}$ is equal to the area of the parallelogram spanned by the projections $\mathbf{v}_0$ and $\mathbf{w}_0$ onto the xy-plane (Figure 21).

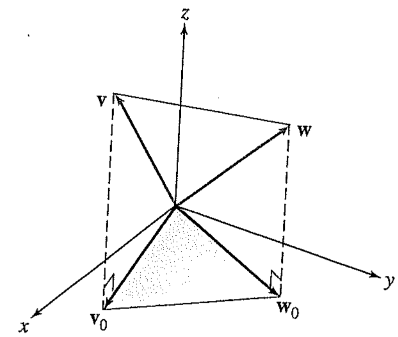

FIGURE 21

56. Formulate and prove analogs of the result in Exercise 55 for the $\mathbf{i}$- and $\mathbf{j}$-components of $\mathbf{v} \times \mathbf{w}$.

57. Show that three points P, Q, R are collinear (lie on a line) if and only if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$.

58. Use the result of Exercise 57 to determine whether the points P, Q, and R are collinear, and if not, find a vector normal to the plane containing them.

(a) $P = (2, 1, 0)$, $Q = (1, 5, 2)$, $R = (-1, 13, 6)$

(b) $P = (2, 1, 0)$, $Q = (-3, 21, 10)$, $R = (5, -2, 9)$

(c) $P = (1, 1, 0)$, $Q = (1, -2, -1)$, $R = (3, 2, -4)$

59. Solve the equation $\langle 1, 1, 1\rangle \times \mathbf{X} = \langle 1, -1, 0\rangle$, where $\mathbf{X} = \langle x, y, z\rangle$. *Note:* There are infinitely many solutions.

60. Explain geometrically why $\langle 1, 1, 1\rangle \times \mathbf{X} = \langle 1, 0, 0\rangle$ has no solution, where $\mathbf{X} = \langle x, y, z\rangle$.

61. Let $\mathbf{X} = \langle x, y, z\rangle$. Show that $\mathbf{i} \times \mathbf{X} = \mathbf{v}$ has a solution if and only if $\mathbf{v}$ is contained in the yz-plane (the $\mathbf{i}$-component is zero).

62. Suppose that vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually orthogonal—that is, $\mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$, and $\mathbf{v} \perp \mathbf{w}$. Prove that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$.

In Exercises 63–66, the **torque** *about the origin O due to a force $\mathbf{F}$ acting on an object with position vector $\mathbf{r}$ is the vector quantity $\tau = \mathbf{r} \times \mathbf{F}$. If several forces $\mathbf{F}_j$ act at positions $\mathbf{r}_j$, then the net torque (units: N-m or lb-ft) is the sum*

$$\tau = \sum \mathbf{r}_j \times \mathbf{F}_j$$

Torque measures how much the force causes the object to rotate. By Newton's Laws, τ is equal to the rate of change of angular momentum.

63. Calculate the torque τ about O acting at the point P on the mechanical arm in Figure 22(A), assuming that a 25-newton force acts as indicated. Ignore the weight of the arm itself.

64. Calculate the net torque about O at P, assuming that a 30-kg mass is attached at P [Figure 22(B)]. The force $\mathbf{F}_g$ due to gravity on a mass m has magnitude $9.8m$ m/s^2 in the downward direction.

- The family of parallel planes with given normal vector $\mathbf{n} = \langle a, b, c \rangle$ consists of planes with equation $ax + by + cz = d$ for some d.
- The plane through three points P, Q, R that are not collinear:
  - $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$
  - $d = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$, where $P = (x_0, y_0, z_0)$
- The intersection of a plane $\mathcal{P}$ with a coordinate plane or a plane parallel to a coordinate plane is called a *trace*. The trace in the yz-plane is obtained by setting $x = 0$ in the equation of the plane (and similarly for the traces in the xz- and xy-planes).

12.5 EXERCISES

Preliminary Questions

1. What is the equation of the plane parallel to $3x + 4y - z = 5$ passing through the origin?

2. The vector $\mathbf{k}$ is normal to which of the following planes?
(a) $x = 1$ (b) $y = 1$ (c) $z = 1$

3. Which of the following planes is not parallel to the plane $x + y + z = 1$?
(a) $2x + 2y + 2z = 1$ (b) $x + y + z = 3$
(c) $x - y + z = 0$

4. To which coordinate plane is the plane $y = 1$ parallel?

5. Which of the following planes contains the z-axis?
(a) $z = 1$ (b) $x + y = 1$ (c) $x + y = 0$

6. Suppose that a plane $\mathcal{P}$ with normal vector $\mathbf{n}$ and a line $\mathcal{L}$ with direction vector $\mathbf{v}$ both pass through the origin and that $\mathbf{n} \cdot \mathbf{v} = 0$. Which of the following statements is correct?
(a) $\mathcal{L}$ is contained in $\mathcal{P}$.
(b) $\mathcal{L}$ is orthogonal to $\mathcal{P}$.

Exercises

In Exercises 1–8, write the equation of the plane with normal vector $\mathbf{n}$ *passing through the given point in the scalar form* $ax + by + cz = d$.

1. $\mathbf{n} = \langle 1, 3, 2 \rangle$, $(4, -1, 1)$
2. $\mathbf{n} = \langle -1, 2, 1 \rangle$, $(3, 1, 9)$
3. $\mathbf{n} = \langle -1, 2, 1 \rangle$, $(4, 1, 5)$
4. $\mathbf{n} = \langle 2, -4, 1 \rangle$, $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$
5. $\mathbf{n} = \mathbf{i}$, $(3, 1, -9)$
6. $\mathbf{n} = \mathbf{j}$, $\left(-5, \frac{1}{2}, \frac{1}{2}\right)$
7. $\mathbf{n} = \mathbf{k}$, $(6, 7, 2)$
8. $\mathbf{n} = \mathbf{i} - \mathbf{k}$, $(4, 2, -8)$

9. Write down the equation of any plane through the origin.

10. Write down the equations of any two distinct planes with normal vector $\mathbf{n} = \langle 3, 2, 1 \rangle$ that do not pass through the origin.

11. Which of the following statements are true of a plane that is parallel to the yz-plane?
(a) $\mathbf{n} = \langle 0, 0, 1 \rangle$ is a normal vector.
(b) $\mathbf{n} = \langle 1, 0, 0 \rangle$ is a normal vector.
(c) The equation has the form $ay + bz = d$
(d) The equation has the form $x = d$

12. Find a normal vector $\mathbf{n}$ and an equation for the planes in Figures 7(A)–(C).

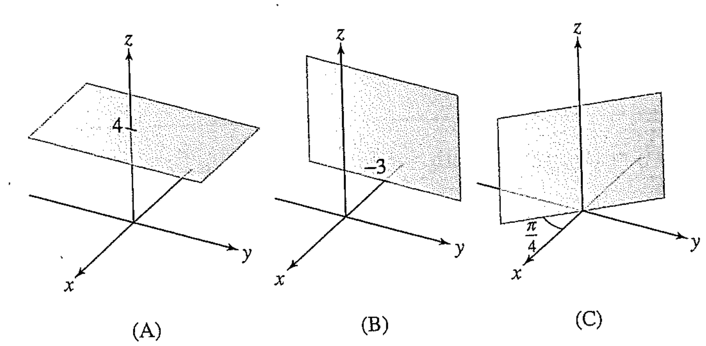

FIGURE 7

In Exercises 13–16, find a vector normal to the plane with the given equation.

13. $9x - 4y - 11z = 2$
14. $x - z = 0$
15. $3(x - 4) - 8(y - 1) + 11z = 0$
16. $x = 1$

In Exercises 17–20, find an equation of the plane passing through the three points given.

17. $P = (2, -1, 4)$, $Q = (1, 1, 1)$, $R = (3, 1, -2)$
18. $P = (5, 1, 1)$, $Q = (1, 1, 2)$, $R = (2, 1, 1)$
19. $P = (1, 0, 0)$, $Q = (0, 1, 1)$, $R = (2, 0, 1)$
20. $P = (2, 0, 0)$, $Q = (0, 4, 0)$, $R = (0, 0, 2)$

In Exercises 21–28, find the equation of the plane with the given description.

21. Passes through O and is parallel to $4x - 9y + z = 3$
22. Passes through $(4, 1, 9)$ and is parallel to $x + y + z = 3$
23. Passes through $(4, 1, 9)$ and is parallel to $x = 3$
24. Passes through $P = (3, 5, -9)$ and is parallel to the xz-plane
25. Passes through $(-2, -3, 5)$ and has normal vector $\mathbf{i} + \mathbf{k}$
26. Contains the lines $\mathbf{r}_1(t) = \langle t, 2t, 3t \rangle$ and $\mathbf{r}_2(t) = \langle 3t, t, 8t \rangle$
27. Contains the lines $\mathbf{r}_1(t) = \langle 2, 1, 0 \rangle + \langle t, 2t, 3t \rangle$ and $\mathbf{r}_2(t) = \langle 2, 1, 0 \rangle + \langle 3t, t, 8t \rangle$

28. Contains $P = (-1, 0, 1)$ and $\mathbf{r}(t) = \langle t+1, 2t, 3t-1 \rangle$

29. Are the planes $\frac{1}{2}x + 2y - z = 5$ and $3x + 12y - 6z = 1$ parallel?

30. Are the planes $2x - 4y - z = 3$ and $-6x + 12y + 3z = 1$ parallel?

In Exercises 31–35, draw the plane given by the equation.

31. $x + y + z = 4$

32. $3x + 2y - 6z = 12$

33. $12x - 6y + 4z = 6$

34. $x + 2y = 6$

35. $x + y + z = 0$

36. Let a, b, c be constants. Which two of the following equations define the plane passing through $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$?

(a) $ax + by + cz = 1$

(b) $bcx + acy + abz = abc$

(c) $bx + cy + az = 1$

(d) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

37. Find an equation of the plane $\mathcal{P}$ in Figure 8.

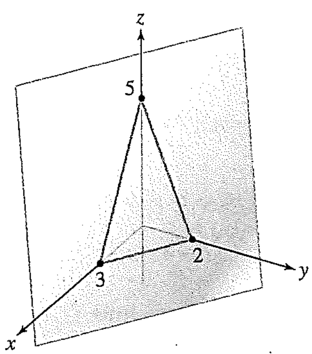

FIGURE 8

38. Verify that the plane $x - y + 5z = 10$ and the line $\mathbf{r}(t) = \langle 1, 0, 1 \rangle + t\langle -2, 1, 1 \rangle$ intersect at $P = (-3, 2, 3)$.

In Exercises 39–42, find the intersection of the line and the plane.

39. $x + y + z = 14$, $\mathbf{r}(t) = \langle 1, 1, 0 \rangle + t\langle 0, 2, 4 \rangle$

40. $2x + y = 3$, $\mathbf{r}(t) = \langle 2, -1, -1 \rangle + t\langle 1, 2, -4 \rangle$

41. $z = 12$, $\mathbf{r}(t) = t\langle -6, 9, 36 \rangle$

42. $x - z = 6$, $\mathbf{r}(t) = \langle 1, 0, -1 \rangle + t\langle 4, 9, 2 \rangle$

In Exercises 43–48, find the trace of the plane in the given coordinate plane.

43. $3x - 9y + 4z = 5$, yz

44. $3x - 9y + 4z = 5$, xz

45. $3x + 4z = -2$, xy

46. $3x + 4z = -2$, xz

47. $-x + y = 4$, xz

48. $-x + y = 4$, yz

49. Does the plane $x = 5$ have a trace in the yz-plane? Explain.

50. Give equations for two distinct planes whose trace in the xy-plane has equation $4x + 3y = 8$.

51. Give equations for two distinct planes whose trace in the yz-plane has equation $y = 4z$.

52. Find parametric equations for the line through $P_0 = (3, -1, 1)$ perpendicular to the plane $3x + 5y - 7z = 29$.

53. Find all planes in $\mathbf{R}^3$ whose intersection with the xz-plane is the line with equation $3x + 2z = 5$.

54. Find all planes in $\mathbf{R}^3$ whose intersection with the xy-plane is the line $\mathbf{r}(t) = t\langle 2, 1, 0 \rangle$.

In Exercises 55–60, compute the angle between the two planes, defined as the angle θ (between 0 and π) between their normal vectors (Figure 9).

55. Planes with normals $\mathbf{n}_1 = \langle 1, 0, 1 \rangle$, $\mathbf{n}_2 = \langle -1, 1, 1 \rangle$

56. Planes with normals $\mathbf{n}_1 = \langle 1, 2, 1 \rangle$, $\mathbf{n}_2 = \langle 4, 1, 3 \rangle$

57. $2x + 3y + 7z = 2$ and $4x - 2y + 2z = 4$

58. $x - 3y + z = 3$ and $2x - 3z = 4$

59. $3(x - 1) - 5y + 2(z - 12) = 0$ and the plane with normal $\mathbf{n} = \langle 1, 0, 1 \rangle$

60. The plane through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ and the yz-plane

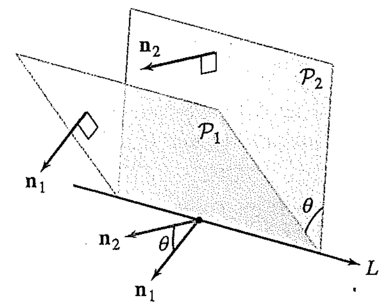

FIGURE 9 By definition, the angle between two planes is the angle between their normal vectors.

61. Find an equation of a plane making an angle of $\frac{\pi}{2}$ with the plane $3x + y - 4z = 2$.

62. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Assume that the planes are not parallel, and let $\mathcal{L}$ be their intersection (a line). Show that $\mathbf{n}_1 \times \mathbf{n}_2$ is a direction vector for $\mathcal{L}$.

63. Find a plane that is perpendicular to the two planes $x + y = 3$ and $x + 2y - z = 4$.

64. Let $\mathcal{L}$ be the intersection of the planes $x + y + z = 1$ and $x + 2y + 3z = 1$. Use Exercise 62 to find a direction vector for $\mathcal{L}$. Then find a point P on $\mathcal{L}$ by *inspection*, and write down the parametric equations for $\mathcal{L}$.

65. Let $\mathcal{L}$ denote the intersection of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. Find parametric equations for the line $\mathcal{L}$. *Hint:* To find a point on $\mathcal{L}$, substitute an arbitrary value for z (say, $z = 2$) and then solve the resulting pair of equations for x and y.

66. Find parametric equations for the intersection of the planes $2x + y - 3z = 0$ and $x + y = 1$.

67. Two vectors $\mathbf{v}$ and $\mathbf{w}$, each of length 12, lie in the plane $x + 2y - 2z = 0$. The angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/6$. This information determines $\mathbf{v} \times \mathbf{w}$ up to a sign ± 1. What are the two possible values of $\mathbf{v} \times \mathbf{w}$?

68. The plane

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{3} = 1$$

intersects the x-, y-, and z-axes in points P, Q, and R. Find the area of the triangle $\triangle PQR$.