1. Determine whether or not the vector field is conservative. If it is, find a function $f$ such that $F = \nabla f$.

$$F(x, y, z) = (3x^2y^3z^3 + yz)\mathbf{i} + (3x^3y^2z^3 + xz)\mathbf{j} + (3x^3y^3z^2 + xy)\mathbf{k}$$

**Sol.:** Here:

$P = 3x^2y^3z^3 + yz$, $Q = 3x^3y^2z^3 + xz$, $R = 3x^3y^3z^2 + xy$.

We have:

$$\frac{\partial P}{\partial y} = 9x^2y^2z^3 + z , \quad \frac{\partial Q}{\partial x} = 9x^2y^2z^3 + z ,$$

these are equal.

$$\frac{\partial P}{\partial z} = 9x^2y^3z^2 + y , \quad \frac{\partial R}{\partial x} = 9x^2y^3z^2 + y ,$$

these are equal.

$$\frac{\partial Q}{\partial z} = 9x^3y^2z^2 + x , \quad \frac{\partial R}{\partial y} = 9x^3y^2z^2 + x ,$$

these are equal.

So $F$ is indeed conservative.

We now need to find a function $f(x, y, z)$ such that

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

Since $\frac{\partial f}{\partial x} = P$,

$$f(x, y, z) = \int (3x^2y^3z^3 + yz)\, dx = x^3y^3z^3 + xyz + g(y, z) ,$$

where $g(y, z)$ is yet-to-be-determined. Using $\frac{\partial f}{\partial y} = Q$,

$$3x^3y^2z^3 + xz + \frac{\partial g}{\partial y} = 3x^3y^2z^3 + xz ,$$

giving

$$\frac{\partial g}{\partial y} = 0 ,$$

So

$$g(y, z) = \int 0\, dy = 0 + h(z) ,$$

where $h(z)$ is yet-to-be-determined. Going back above:

$$f(x, y, z) = x^3y^3z^3 + xyz + h(z) ,$$
Using $\frac{\partial f}{\partial z} = R$, we get:

$$3x^3y^3z^2 + xy + h'(z) = 3x^3y^3z^2 + xy,$$

giving $h'(z) = 0$ so $h(z) = C$ and we have

$$f(x, y, z) = x^3y^3z^3 + xyz + C,$$

but we can forget about the $C$ (we were asked to find a potential function, not all of them), so

**Ans. to second part:** $f(x, y, z) = x^3y^3z^3 + xyz$.

2. Evaluate

$$\int_C 5y \, dx + 10x \, dy,$$

where $C$ is the closed curve consisting of the boundary of the rectangle

$$\{ (x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1 \}.$$

**Sol.:** We are supposed to use **Green’s Theorem**.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

Since no orientation is mentioned we take the default one of positive (counter-clockwise).

In this problem $P = 5y$ and $Q = 10x$, so we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10 - 5 = 5,$$

and the vector-field line-integral that we have to compute equals:

$$\int \int_D 5 \, dA = 5 \int \int_D 1 \, dA = 5 \text{Area}(D),$$

(since an area integral with the integrand being 1 equals the area). By elementary-school geometry, the area of $D$ is $1 \cdot 1 = 1$, so we have:

**Ans.:** 5 (type number).

**Comments:** Everyone got it right!