MATH 251 (1-3,10), Dr. Z. , Solutions to Exam 2, Thurs., Nov. 19, 2009, 10:20-11:40am, SEC 118

1. (10 pts.) (i) Prove that

$$\mathbf{F} = \langle 2e^{2x+3y+5z} + y^2 + 1 \, , \, 3e^{2x+3y+5z} + 2xy + 1 \, , \, 5e^{2x+3y+5z} \rangle$$

is a conservative vector field.

(ii) Find a function $f(x, y, z)$ such that $\mathbf{F} = \bigtriangledown f$.

(iii) Compute $f(1, 1, 1) - f(0, 0, 0) - e^{10} + 1$

Ans. to (iii): 3

Sol.

(i) Here

$$F_1 = 2e^{2x+3y+5z} + y^2 + 1 \, , \, F_2 = 3e^{2x+3y+5z} + 2xy + 1 \, , \, F_3 = 5e^{2x+3y+5z}$$

$$\frac{\partial F_1}{\partial y} = 6e^{2x+3y+5z} + 2y \quad , \quad \frac{\partial F_2}{\partial x} = 6e^{2x+3y+5z} + 2y \quad .$$

$$\frac{\partial F_1}{\partial z} = 10e^{2x+3y+5z} \quad , \quad \frac{\partial F_3}{\partial x} = 10e^{2x+3y+5z} \quad .$$

$$\frac{\partial F_2}{\partial z} = 15e^{2x+3y+5z} \quad , \quad \frac{\partial F_3}{\partial y} = 15e^{2x+3y+5z} \quad .$$

So

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad , \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad , \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad ,$$

and $\mathbf{F}$ is indeed conservative.

(ii) Since $\frac{\partial f}{\partial x} = 2e^{2x+3y+5z} + y^2 + 1$, we have

$$f(x, y, z) = \int (2e^{2x+3y+5z} + y^2 + 1) \, dx = e^{2x+3y+5z} + xy^2 + x + g(y, z) \quad ,$$

where $g(y, z)$ is yet-to be found. Taking derivatives with respect to y:

$$\frac{\partial f}{\partial y} = 3e^{2x+3y+5z} + 2xy + \frac{\partial g}{\partial y} \quad .$$

But we know that $\frac{\partial f}{\partial y} = 3e^{2x+3y+5z} + 2xy + 1$, so we have

$$3e^{2x+3y+5z} + 2xy + \frac{\partial g}{\partial y} = 3e^{2x+3y+5z} + 2xy + 1 \quad .$$

Doing the algebra we get

$$\frac{\partial g}{\partial y} = 1 \quad .$$

Integrating with respect to y we get

$$g(y,z) = \int 1\, dy = y + h(z) \quad ,$$

where $h(z)$ is yet-to-be-found. By **back-substitution** we have:

$$f(x,y,z) = e^{2x+3y+5z} + xy^2 + x + y + h(z) \quad .$$

Finally, take partial-derivative with resepct to z getting

$$f_z = 5e^{2x+3y+5z} + h'(z) = 5e^{2x+3y+5z}$$

that gives $h'(z) = 0$ so $h(z) = C$ (constant), that we can ignore (i.e. take to be 0). So

$$f(x,y,z) = e^{2x+3y+5z} + xy^2 + x + y \quad .$$

(iii)

$$f(1,1,1) - f(0,0,0) - e^{10} + 1 = e^{10} + 1 \cdot 1^2 + 1 + 1 - (e^0 + 0 \cdot 0^2 + 0 + 0) - e^{10} + 1 = 3 \quad .$$

Ans. to (iii) : 3 (type number).

2. (10 points) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \quad ,$$

$$\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + \mathbf{k} \quad , \quad 0 \leq t \leq 1 \quad .$$

Ans.: 1 (type number).

First Solution: The curve is $x = t^3, y = t^2, z = 1$, $(0 \leq t \leq 1)$ or in vector-notation:

$$\mathbf{r}(t) = \langle t^3, t^2, 1 \rangle \quad ,$$

So

$$\mathbf{r}'(t) = \langle 3t^2, 2t, 0 \rangle \quad .$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle yz, xz, xy \rangle \cdot \langle 3t^2, 2t, 0 \rangle dt$$

$$= \int_0^1 \langle yz, xz, xy \rangle \cdot \langle 3t^2, 2t, 0 \rangle dt = \int_0^1 \langle (t^2)(1), (t^3)(1), (t^3)(t^2) \rangle \cdot \langle 3t^2, 2t, 0 \rangle dt$$

$$= \int_0^1 (3t^4 + 2t^4 + 0)\, dt = \int_0^1 (5t^4)\, dt = t^5 \Big|_0^1 = 1 \quad .$$

Second Solution: This vector-field is conservative (check!), and it is easy to see by inspection that $f(x, y, z) = xyz$ is the potential function (check that $\bigtriangledown f = \mathbf{F}$). So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(End) - f(Start) = f(1, 1, 1) - f(0, 0, 0) = 1 - 0 = 1 \quad .$$

Comments: To get full credit you only need to do **one** of the above, of course.

3. (10 points) Evaluate

$$\int\int\int_E \frac{9}{\pi}(x^2+y^2)^3\,dV \quad ,$$

where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.

Ans.: 4

Sol.: We need to use **cylindrical coordinates**. The "floor" is the circle $r=1$ and the ceiling is $z^2=4r^2$ which could mean $z=2r$ or $z=-2r$, but since the floor is the xy-plane, obviously we take $z=2r$. Remember that in cylindrical (and polar), $x^2+y^2=r^2$, and recall that $dV=rdzdrd\theta$. The volume-integral of the problem is $\frac{9}{\pi}$ times

$$\int_0^{2\pi}\int_0^1\int_0^{2r}(r^2)^3\,rdzdrd\theta=\int_0^{2\pi}\int_0^1\int_0^{2r}r^7dzdrd\theta \quad .$$

The **inner integral** is:

$$\int_0^{2r} r^7\,dz = r^7(2r)=2r^8 \quad .$$

The **middle-integral** is:

$$\int_0^1 2r^8\,dr = 2\frac{r^9}{9}\Big|_0^1=\frac{2}{9} \quad .$$

The outside integral

$$\int_0^{2\pi}\frac{2}{9}\,d\theta=\frac{4\pi}{9} \quad .$$

Now multiply by $\frac{9}{\pi}$ to get that the answer is

$$\frac{4\pi}{9}\cdot\frac{9}{\pi}=4 \quad .$$

4. (10 points) Evaluate the iterated integral

$$\int_0^1 \int_x^{2x} \int_0^{x+y} \frac{24z}{19}\, dz\, dy\, dx \quad .$$

Ans.: 1

Sol.: Let's take the annoying $\frac{24}{19}$ out of the integration (we get back to it at the end). The inner integral is

$$\int_0^{x+y} z\, dz = \frac{z^2}{2}\Big|_0^{x+y} = \frac{(x+y)^2}{2} \quad .$$

The **middle integral** is

$$\int_x^{2x} \frac{(x+y)^2}{2}\, dy = \frac{(x+y)^3}{6}\Big|_{y=x}^{y=2x} = \frac{(x+2x)^3}{6} - \frac{(x+x)^3}{6} = \frac{(3x)^3}{6} - \frac{(2x)^3}{6} = \frac{27x^3}{6} - \frac{8x^3}{6} = \frac{19x^3}{6} \quad .$$

The outside integral is

$$\int_0^1 \frac{19x^3}{6} = \frac{19x^4}{6 \cdot 4}\Big|_0^1 = \frac{19}{24} \quad .$$

Finally, multiplying by the constant in front, we have

$$\frac{24}{19} \cdot \frac{19}{24} = 1 \quad .$$

Comment: Most people, after the first step, expanded $(x+y)^2$ as $x^2+2xy+y^2$ and made the problem longer than it should be. It is very useful to memorize the integral of a generalized power

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} \quad .$$

(Of course, here we need

$$\int (ay+b)^n\, dy = \frac{(ay+b)^{n+1}}{a(n+1)} \quad .$$

with $a=1$ and $b=x$.)

5. (10 points) Use the given transformation to evaluate the integral

$$\int\int_R 20(2x+y)^2\,dA \quad ,$$

where R is the triangular region with vertices $(0,0)$,$(2,-3)$, $(3,-5)$; $x = 3u - v$, $y = -5u + 2v$.

Ans.: 5 .

Sol.: First, we need to find the counterparts of the three vertices:
For $(0,0)$, $x = 0, y = 0$ so we have to solve

$$3u - v = 0 \quad , -5u + 2v = 0 \quad .$$

whose solution is $u = 0, v = 0$, and so the point in the uv-plane corresponding to $(x,y) = (0,0)$ is $(0,0)$.
For $(2,-3)$, $x = 2, y = -3$ so we have to solve

$$3u - v = 2 \quad , -5u + 2v = -3 \quad .$$

From the first equationL $v = 3u - 2$, plugging into the second, we get $-5u + 2(3u-2) = -3$ which is $u - 4 = -3$ which gives $u = 1$, and by **back substitution** $v = 3(1) - 2 = 1$, and so the point in the uv-plane corresponding to $(x,y) = (2,-3)$ is $(u,v) = (1,1)$.
For $(3,-5)$, $x = 3, y = -5$ so we have to solve

$$3u - v = 3 \quad , -5u + 2v = -5 \quad .$$

From the first equation $v = 3u - 3$, plugging into the second, we get $-5u + 2(3u-3) = -5$ which is $u - 6 = -5$ which gives $u = 1$, and by **back substitution** $v = 3(1) - 3 = 0$, and so the point in the uv-plane corresponding to $(x,y) = (3,-5)$ is $(u,v) = (1,0)$.
So the triangle in the (u,v)-plane is much simpler. It has vertices $(0,0),(1,0),(1,1)$ that in type-I notation is:

$$\{(u,v)\,|\,0 \le u \le 1 \quad , \quad 0 \le v \le u\} \quad .$$

Next we have to compute the **Jacobian**, J:

$$J = (x_u)(y_v) - (x_v)(y_u) = (3)(2) - (-1)(-5) = 1 \quad .$$

We now convert the xy-integral into the uv-language:

$$\int\int_{R'} 20\left(2(3u-v) + (-5u+2v)\right)^2 |J|\,dA = \int_0^1 \int_0^u 20(u)^2 dv du \quad .$$

The inner integral is $20u^3$ and the outer integral is $\int_0^1 20u^3 = 5u^4\Big|_0^1 = 5$.
Comment: Some people forgot to worry about the Jacobian. They still got the right answer, since, by accident, the Jacobian happened to be 1 in this problem. Nevertheless they lost 3 points.

6. (10 points) Evaluate the iterated integral by converting to polar coordinates.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{27}{64\pi} (x^2 + y^2)^2 \, dy \, dx$$

Ans.: 9 .

Sol. Use polar coordinates. Recall that $x^2 + y^2$ in polar language is r^2, so the integrand is r^4.
The region is the inside **full** circle of radius 2, center the origin, so a polar description of the region is:

$$\{(r, \theta) \,|0 \leq r \leq 2 \,,\, 0 \leq \theta \leq 2\pi\} \quad .$$

Also $dA = r dr d\theta$, so our integral (ignoring the annoying constant $\frac{27}{64\pi}$ for now) is:

$$\int_{0}^{2\pi} \int_{0}^{2} r^4 r \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{2} r^5 \, dr \, d\theta \quad .$$

The inside integral is $\frac{r^6}{6}\Big|_0^2 = \frac{64}{6} = \frac{32}{3}$. The outside integral is $\int_0^{2\pi} \frac{32}{3} \, d\theta = \frac{64\pi}{3}$. Finally, multiplying by the annoying $\frac{27}{64\pi}$ we get that the answer is:

$$\frac{27}{64\pi} \cdot \frac{64\pi}{3} = 9 \quad .$$

7. (10 points) Let A be the **number**

$$A = \int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy \quad .$$

What is $1 + e^4 - A$? (Hint: Not even Dr. Z. can do $\int e^{x^2}\, dx$, so you must be clever, and first change the order of integration.)

Ans.: 2

Sol. The integral is in type II format:

$$\{(x, y) | 0 \leq y \leq 4\, ,\, y/2 \leq x \leq 2\}$$

If you draw it (do it!), you get a triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$. A type I description of the same region is:

$$\{(x, y) | 0 \leq x \leq 2\, ,\, 0 \leq y \leq 2x\}$$

So the type I formulation of the integral is:

$$A = \int_0^2 \int_0^{2x} e^{x^2}\, dydx \quad .$$

The **inner integral** is

$$\int_0^{2x} e^{x^2}\, dy = e^{x^2} \int_0^{2x} dy = e^{x^2}(2x) \quad .$$

The **outer integral** is: (do the substitution $u = x^2$)

$$A = \int_0^2 2xe^{x^2} = e^{x^2}\Big|_0^2 = e^4 - e^0 = e^4 - 1 \quad .$$

Finally,

$$1 + e^4 - A = 1 + e^4 - (e^4 - 1) = 2 \quad .$$

Comment: Quite a few people only computed A and left the final answer as $e^4 - 1$. Please read the **whole** question. Being able to follow instructions is much more important than multivariable calculus.

8. (10 points) Calculate the double integral

$$\int\int_R \frac{27x^2y^2}{(\ln 2)(x^3+1)}\, dA \quad ,$$

$$R = \{(x,y)\,|\,0 \le x \le 1\,,\,-1 \le y \le 1\,\} \quad .$$

Ans.: 6 .

Sol. : The iterated integral is

$$\frac{27}{\ln 2}\int_0^1 \int_{-1}^1 \frac{27x^2y^2}{(\ln 2)(x^3+1)}\, dydx =$$

The integrand is **separable**, and the region is a **rectangle** so we have:

$$\frac{27}{\ln 2}(\int_0^1 \frac{x^2}{x^3+1}\, dx)(\int_{-1}^1 y^2\, dy) \quad .$$

The first integral is almost of the form TOP/BOT where the TOP is the derivative of BOT, and whose integral is $\ln|BOT|$.

$$\int_0^1 \frac{x^2}{x^3+1}\, dx = \frac{1}{3}\int_0^1 \frac{3x^2}{x^3+1}\, dx = \frac{1}{3}\int_0^1 \frac{3x^2}{x^3+1}\, dx = \frac{1}{3}\ln|x^3+1|\big|_0^1 = \frac{1}{3}(\ln 2 - \ln 1) = \frac{\ln 2}{3} \quad .$$

The second integral is even simpler:

$$\int_{-1}^1 y^2\, dy = \frac{y^3}{3}\Big|_{-1}^1 = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3} \quad .$$

Combining, we get that the answer is:

$$\frac{27}{\ln 2}\cdot\frac{\ln 2}{3}\cdot\frac{2}{3} = 6 \quad .$$

9. (10 points) Use Lagrange multipliers to find the maximum value of the function $f(x, y) = x^2y - 27$ subject to the constraint $x + y = 6$.

Ans.: 5 .

Sol. The goal function is $f(x, y) = x^2y - 27$. The constraint function is $g(x, y) = x + y - 6$.

$$\bigtriangledown f = \langle 2xy \, , \, x^2 \rangle \quad , \quad \bigtriangledown g = \langle 1 \, , \, 1 \rangle$$

We have to solve $\bigtriangledown f = \lambda \bigtriangledown g$. In this problem it is:

$$\langle 2xy \, , \, x^2 \rangle = \lambda \langle 1 \, , \, 1 \rangle \quad .$$

Spelling it out we have the equations

$$2xy = \lambda \quad , x^2 = \lambda$$

In addition we have the **constraint** equation $x + y = 6$. It is always good to get rid of λ. Subtracting the first from the second, we get

$$2xy - x^2 = 0$$

Factorizing, we get $x(2y - x) = 0$ so we have $x = 0$ or $x = 2y$. The first option gives the solution $(x, y) = (0, 6)$, and the second one gives $2y + y = 6$ so $3y = 6$ so $y = 2$ and $x = 2 \cdot 2 = 4$, yielding $(x, y) = (4, 2)$.
Now it is time to plug-in. $f(0, 6) = -27$ and $f(4, 2) = 4^2 \cdot 2 - 27 = 32 - 27 = 5$. So the **minimum value** is -27 and the **maximum value** is 5. But no one asked us about the minimum value, so don't mention it.

10. (10 points) Find the local maximum and minimum **values**, and saddle point(s) of the function $f(x,y) = x^4 + y^4 - 4xy + 5$.

Local maximum value(s): None

Local minimum value(s) 3 (at $(1,1)$ and $(-1,-1)$)

saddle point(s): $(0,0)$

Sol.:

$$f_x = 4x^3 - 4y \quad , \quad f_y = 4y^3 - 4x \quad ,$$

For future reference

$$f_{xx} = 12x^2 \quad , \quad f_{xy} = -4 \quad , \quad f_{yy} = 12y^2 \quad .$$

We must first solve the system of two equations and two unknowns: $f_x = 0, f_y = 0$.

$$4x^3 - 4y = 0 \quad , \quad 4y^3 - 4x = 0 \quad .$$

So $y = x^3$ $x = y^3$. Substituting the first equation into the second one gives

$$x = (x^3)^3 = x^9$$

So

$$x^9 - x = 0$$

Factoriong:

$$x(x^8 - 1) = 0$$

giving $x = 0, x = -1, x = 1$. But $y = x^3$ so when $x = 0, y = 0^3 = 0$, when $x = 1$, $y = 1^3 = 1$ and when $x = -1$, $y = (-1)^3 = -1$. So the three critical points are $(-1,-1)$, $(0,0)$, and $(1,1)$.

Now it is time to examine each point according to its merit.

When $(x,y) = (0,0)$, we have $f_{xx} = 0, f_{xy} = -4, f_{yy} = 0$, so $D = (0)(0) - (-4)^2 = -16 < 0$. Since D is **negative**, $(0,0)$ is a **saddle point**.

When $(x,y) = (1,1)$, we have $f_{xx} = 12, f_{xy} = -4, f_{yy} = 12$, so $D = (12)(12) - (-4)^2 = 144 - 16 = 128 > 0$. Since D is **positive**, it is either a max or a min. Since f_{xx} is positive, this is a **local min**. The local min value is $f(1,1) = 1^4 + 1^4 - 4(1)(1) + 5 = 3$.

When $(x,y) = (-1,-1)$, we have $f_{xx} = 12, f_{xy} = -4, f_{yy} = 12$, so $D = (12)(12) - (-4)^2 = 144 - 16 = 128 > 0$. Since D is **positive**, it is either a max or a min. Since f_{xx} is positive, this is a **local min**. The local min value is $f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 5 = 3$.

Comment: Some people wrote for the value, 128. This is **nonsense**. The min value is obtained by plugging-in the minimum point into $f(x,y)$. The actual value of the discriminant D is not important, only whether it is positive or negative.