Problem Type 17.2a: Use Stokes’ Theorem to evaluate $\int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$ ,

and $S$ is some surface with a given orientation (that should boil down to either outwards or inwards).

Example Problem 17.2a: Use Stokes’ Theorem to evaluate $\int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = x^2 e^{2yz} \mathbf{i} + y^2 e^{3xz} \mathbf{j} + z^2 e^{4xy} \mathbf{k}$$ ,

and $S$ is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented upwards.
1. You are going to use Stokes’ Theorem
\[ \int \int \text{curl} \, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}. \]

The challenge is to find the bounding curve \( C \).

For hemispheres \( x^2 + y^2 + z^2 = R^2, \ z \geq 0 \), like in this problem, the bounding curve is simply the circle \( x^2 + y^2 = R^2 \), on the \( xy \)-plane \( z = 0 \) and the parametric representation is
\[ x = R \cos t, \ y = R \sin t, \ z = 0. \]

If it is the hemisphere \( z \leq 0 \) then it is the same but in the negative direction. If it is the hemisphere \( x^2 + y^2 + z^2 = R^2, \ y \leq 0 \), then the parametric representation is
\[ x = R \cos t, \ z = R \sin t, \ y = 0. \]

etc.

If it is the part of a surface \( z = g(x, y) \) that lies above a plane \( z = a \), oriented outwards, then \( C \) is obtained by solving \( g(x, y) = a \) and representing \( g(x, y) = a \) in parametric notation and adding to it \( z = a \). The orientation of \( C \) is such as to obey the right-hand rule.

At the end you need to represent \( C \) in parametric form
\[ \mathbf{r}(t) = (x(t), y(t), z(t)), \ a \leq t \leq b, \]
for some expressions \( x(t), y(t), z(t) \) of \( t \) and some numbers \( a \) and \( b \).
2. Plug in the expressions for \(x, y, z\) in terms of \(t\) in \(F\), in order to express it in terms of the parameter \(t\). Also figure out
\[
dr = r'(t) \, dt.
\]

\[
F(x, y, z) = (3 \cos t)^2 e^0 i + (3 \sin t)^2 e^0 j + 0 k
\]
\[
= 9 \cos^2 t \, i + 9 \sin^2 t \, j + 0 k,
\]
\[
dr(t) = (-3 \sin t \, i + 3 \cos t \, j + 0 k) \, dt.
\]

3. Find the dot product \(F \cdot dr\) and integrate from \(t = a\) to \(t = b\).

\[
F \cdot dr = ((9 \cos^2 t) \cdot (-3 \sin t) + (9 \sin^2 t) \cdot (3 \cos t) + 0) \, dt =
\]
\[
(-27 \cos^2 t \sin t + 27 \sin^2 t \cos t) \, dt.
\]
Finally, integrating from \(t = 0\) to \(t = 2\pi\), we get
\[
\int_C F \cdot dr = \int_0^{2\pi} (-27 \cos^2 t \sin t + 27 \sin^2 t \cos t) \, dt
\]
\[
= 9 \cos^3 t - 9 \sin^3 t \bigg|_0^{2\pi}
\]
\[
= (9 \cos^3(2\pi) + 9 \sin^3(2\pi)) - (9 \cos^3(0) + 9 \sin^3(0))
\]
\[
= 9 - 9 = 0.
\]

Ans.: 0.

**Problem Type 17.2b**: Use Stokes’ Theorem to evaluate \(\int \int_S \text{curl} \, F \cdot dS\), if
\[
F = P(x, y, z) \, i + Q(x, y, z) \, j + R(x, y, z) \, k,
\]
and \(S\) consists of the top and four sides (but not the bottom) of the cube with vertices \((\pm A, \pm A, \pm A)\).

**Example Problem 17.2b**: Use Stokes’ Theorem to evaluate \(\int \int_S \text{curl} \, F \cdot dS\), if
\[
F = x^2 y^2 \, i + y^2 z \, j + yz^2 \, k,
\]
and \(S\) consists of the top and four sides (but not the bottom) of the cube with vertices \((\pm 2, \pm 2, \pm 2)\).
1. In this problem, it is possible to find the bounding curve $C$, and use Stokes’s Theorem directly, but, in this case $C$ is a square with four sides and we would have to do four integrals, and it is a pain. We will use Stokes’s theorem indirectly by finding another surface with the same bounding curve. Naturally for a box in which the given surface consists of the top and the four walls, the bottom is such a surface.

2. Find $\text{curl} \, \mathbf{F}$.

$$\text{curl} \, \mathbf{F} = (z^2 - y^2) \mathbf{i} + 0 \mathbf{j} - 2x^2 y \mathbf{k}$$

(You do it!)

3. Plug-in $z = -A$ and note that $dS = dxdy \mathbf{k}$, and the region of integration is

$$\{(x,y) \mid -A \leq x \leq A, -A \leq y \leq A\}$$

Do the integration

3. When $z = -2$,

$$\text{curl} \, \mathbf{F} = (4 - y^2) \mathbf{i} + 0 \mathbf{j} - 2x^2 y \mathbf{k}$$

So

$$\text{curl} \, \mathbf{F} \cdot d\mathbf{S} = ((4-y^2) \mathbf{i} + 0 \mathbf{j} - 2x^2 y \mathbf{k}) \cdot \mathbf{k} = -2x^2 y$$

Finally,

$$\int \int_S \text{curl} \, \mathbf{F} \cdot d\mathbf{S} = \int_{-2}^{2} \int_{-2}^{2} -2x^2 y \, dx \, dy$$

$$= \int_{-2}^{2} \left[ \int_{-2}^{2} -2x^2 y \, dx \right] \, dy = \int_{-2}^{2} (-2y) \left[ \frac{x^3}{3} \right]_{-2}^{2} \, dy$$

$$= \int_{-2}^{2} (-2y) \frac{16}{3} \, dy$$

$$= -\frac{32}{3} \int_{-2}^{2} y \, dy = -\frac{32}{3} \left[ \frac{y^2}{2} \right]_{-2}^{2} = -\frac{32}{3} \cdot 0 = 0$$

Ans.: 0.
Problem Type 17.2c: Use Stokes’ Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k},$$

and $C$ is a curve that bounds some surface (that you have to figure out!), $z = g(x, y)$, above the region $\{(x, y) | (x, y) \in D\}$ (that you have to find!). $C$ is oriented counterclockwise as viewed from above.

Example Problem 17.2c: Use Stokes’ Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = (2x + y^2) \mathbf{i} + (2y + z^2) \mathbf{j} + (3z + x^2) \mathbf{k},$$

and $C$ is the triangle with vertices $(2, 0, 0), (0, 2, 0), (0, 0, 2)$, and is oriented counterclockwise as viewed from above.

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**Steps**

1. Find a convenient surface that our curve bounds, and express it in terms of $z = g(x, y)$. Also figure out its projection on the $xy$-plane.

2. Find $\text{curl} \mathbf{F}$.

**Example**

1. The three vertices of our triangle lie on the plane $x + y + z = 2$ (you do it!), so $z = 2 - x - y$, and $g(x, y) = 2 - x - y$. Also the projection of the triangle on the $xy$ plane is bounded by the line $x + y = 2$ and the axes, so it is the type I region

$$D = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 2-x\}.$$

2. $\text{curl} \mathbf{F} = -2z \mathbf{i} - 2x \mathbf{j} - 2y \mathbf{k}$ (You do it!)
3. You have to use Stokes’ Theorem

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \]

Convert the surface integral into an area integral using the formula from 16.7

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA \]

Be also sure to replace \( z \) by \( g(x, y) \). **Note:** The \( \mathbf{F} \) from this formula is **not** the same as our \( \mathbf{F} \), it is rather our \( \text{curl} \mathbf{F} \), so use \( \mathbf{F} \) as a local variable.

3.

\[ \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \int_S (-2z \mathbf{i} - 2x \mathbf{j} - 2y \mathbf{k}) \cdot d\mathbf{S} \]

where the surface is the one from step 1, i.e. \( z = 2 - x - y \) over \( D \) given above. Here we have \( P = -2z \), \( Q = -2x \), \( R = -2y \), \( g = 2 - x - y \), so

\[ \int \int_D (-2z)(-1) - (-2x)(-1) + (-2y)) dA \]

Now we have to replace \( z \) by \( 2 - x - y \), so this equals

\[ \int \int_D (-2(2-x-y)-2x-2y) dA = \int \int_D -4 dA \]

\[ = -4 \int \int_D dA \]

where \( D \) is the triangle

\[ D = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 2-x \} \]

In this case the area integral is simply the area of the triangle \((2 \cdot 2)/2 = 2\) times \(-4\), so the answer is \(-8\), but of course you are welcome to to do it without the shortcut:

\[ \int \int_D (-4) dA = \int_0^2 \int_0^{2-x} (-4) \, dx \, dy = -8 \]

**Ans.:** \(-8\).

**A Problem from a Previous Final**

By using Stokes’ Theorem, or otherwise, evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where

\[ F(x, y, z) = yz^2 \mathbf{i} + xz^2 \mathbf{j} + 2xyz \mathbf{k} \]

where \( C \) is the curve if intersection of the plane \( x + y + z = 1 \) and the cylinder \( x^2 + y^2 = 9 \), oriented counterclockwise as viewed from above. Be sure to explain everything.

**Ans.:** \( 0 \).
Another Problem from a Previous Final

By using Stokes' theorem, or otherwise, evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$,

where $\mathbf{F}$ is the vector field

$$\mathbf{F}(x, y, z) = \left\langle 2xy^2z^2, 2x^2yz^2, 2x^2y^2z \right\rangle,$$

and $C$ is the closed curve going from $(1, 0, 1)$ to $(3, 4, 9)$, and then from $(3, 4, 9)$ to $(-1, 4, 11)$, and then from $(-1, 4, 11)$ to $(5, 2, 11)$ and finally from $(5, 2, 11)$ back to the starting point $(1, 0, 1)$. Explain everything!

Ans.: 0.