Problem Type 17.1aa: Use Green’s Theorem to evaluate the line integral along the given positively oriented curve.

\[ \int_C P(x, y) \, dx + Q(x, y) \, dy , \]

where \( C \) is a given curve.

Example Problem 17.1aa: Use Green’s Theorem to evaluate the line integral along the given positively oriented curve.

\[ \int_C \frac{2x^2y^3}{3} \, dx + 4xy^3 \, dy , \]

where \( C \) is the triangle with vertices (0, 0), (1, 3), and (0, 3).

Steps

1. Set-up Green’s Theorem

\[ \int_C P(x, y) \, dx + Q(x, y) \, dy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA , \]

where \( D \) is the region enclosed by \( C \).

Example

1. Here \( P = \frac{2x^2y^3}{3} \), \( Q = 4xy^3 \), so

\[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x} (4xy^3) - \frac{\partial}{\partial y} \left( \frac{2x^2y^3}{3} \right) = 4y^3 - 2x^2y^2 . \]

We have to evaluate the area integral

\[ \int \int_D (4y^3 - 2x^2y^2) \, dA , \]

where \( D \) is the region inside our triangle.
2. Draw $D$, and write it as a type I or
type II region. Set up our area integral as
an iterated integral.

2. Our triangle has one side along the $y$-
axis, so it is more convenient to express
it as a type II region. The hypotenuse is
the line $y = 3x$, that should be written as
$x = y/3$, since **now** $y$ is the boss! So

$$D = \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq y/3\}.$$ 

The iterated integral is thus:

$$\int_0^3 \int_0^{y/3} (4y^3 - 2x^2y^2) \, dx \, dy.$$ 

3. Evaluate this iterated integral.

3. The inner integral is

$$\int_0^{y/3} (4y^3 - 2x^2y^2) \, dx = 4y^3x - 2y^2x^3\bigg|_{x=0}^{x=y/3}$$

$$= 4y^3(y/3) - 2 \frac{y^2}{3}(y/3)^3 = \frac{4}{3}y^4 - \frac{2}{81}y^5.$$ 

The whole thing is:

$$\int_0^3 \left[ \int_0^{y/3} (4y^3 - 2x^2y^2) \, dx \right] \, dy,$$

$$= \int_0^3 \left( \frac{4}{3}y^4 - \frac{2}{81}y^5 \right) \, dy = \frac{4}{3} \frac{y^5}{5} - \frac{2}{81} \frac{y^6}{6}\bigg|_{y=0}^{y=3}$$

$$= \frac{4}{3} \frac{3^5}{5} - \frac{2}{81} \frac{3^6}{6} - 0 = \frac{309}{5}.$$ 

**Ans.:** $\frac{309}{5}$.

**Problem Type 17.1ab**: Use Green’s Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, (Check orientation of the
curve before applying the theorem).

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle,$$

where $C$ is a certain given closed curve.

**Example Problem 17.1ab**: Use Green’s Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, (Check orientation of
the curve before applying the theorem).

$$\mathbf{F}(x, y) = \langle \sin x + y^2, x + \cos^3 y \rangle.$$
where $C$ consists of the arc of the curve $y = \sin x$ from $(0,0)$ to $(\pi,0)$ and the line segment from $(\pi,0)$ to $(0,0)$.

**Steps**

1. **Set-up Green’s Theorem**

   \[
   \int_C P(x,y) \, dx + Q(x,y) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA ,
   \]

   where $D$ is the region enclosed by $C$. Also decide whether the specified description of the curve is in the positive direction (counterclockwise) or in the negative direction (clockwise).

2. **Draw $D$, and write it as a type I or type II region. Set up our area integral as an iterated integral.**

**Example**

1. Here $P = \sin x + y^2$, $Q = x + \cos^3 y$, so

   \[
   \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(x + \cos^3 y) - \frac{\partial}{\partial y}(\sin x + y^2) = 1 - 2y .
   \]

   We have to evaluate the area integral

   \[
   \int \int_D (1 - 2y) \, dA ,
   \]

   where $D$ is the region inside our closed curve. The way $C$ is described is clockwise so it is in the negative direction. So at the end, we have to take the negative of the result. So multiplying by $(-1)$, we really have to evaluate

   \[
   \int \int_D (2y - 1) \, dA .
   \]

2. Our region has one side along the $x$-axis, from $x = 0$ to $x = \pi$, and the other part along the curve $y = \sin x$. It is more convenient to express it as a type I region.

   \[
   D = \{(x,y) \mid 0 \leq x \leq \pi, \, 0 \leq y \leq \sin x \} .
   \]

   The iterated integral is thus:

   \[
   \int_0^\pi \int_0^{\sin x} (2y - 1) \, dy \, dx .
   \]
3. Evaluate this iterated integral.

\[\int_0^{\sin x} (2y-1) \, dy = y^2 - y \bigg|_0^{\sin x} = \sin^2 x - \sin x \, .\]

The whole thing is:

\[\int_0^{\pi} \left[ \int_0^{\sin x} (2y-1) \, dy \right] \, dx \, ,\]

\[= \int_0^{\pi} (\sin^2 x - \sin x) \, dx \]

\[= \int_0^{\pi} \left( \frac{1 - \cos 2x}{2} - \sin x \right) \, dx \]

\[= \frac{x}{2} - \frac{\sin 2x}{4} + \cos x \bigg|_0^{\pi} \]

\[= \left( \frac{\pi}{2} - \frac{\sin 2\pi}{4} + \cos \pi \right) - \left( \frac{0}{2} - \frac{\sin (2 \cdot 0)}{4} + \cos 0 \right) \]

\[= \frac{\pi}{2} - 0 - 1 - (0 - 0 + 1) = \frac{\pi}{2} - 2 \, .\]

Ans.: \(\frac{\pi}{2} - 2\).

A Problem from a previous Final

Evaluate

\[\int_C (5y - \sin(e^x)) \, dx + (10x - e^{\cos^2 y}) \, dy \, ,\]

where \(C\) is the closed curve consisting of the boundary of the rectangle

\[\{ (x, y) \mid 0 \leq x \leq 4 \, , \, 0 \leq y \leq 3 \}.\]

Ans. 60.