Problem Type 16.5a: Evaluate the surface integral

\[ \int \int_S F(x, y, z) \, dS \]

where \( S \) is a given surface that is either given parametrically in terms of \( u \, v \), or can be made so, or can be written in the form \( z = f(x, y), \{(x, y) | (x, y) \in D\} \) for some set \( D \) in the \( xy \)-plane.

Example Problem 16.5a: Evaluate the surface integral

\[ \int \int_S 2x^2z^2 \, dS \]

where \( S \) is the part of the cone \( z^2 = x^2 + y^2 \) that lies between the planes \( z = 1 \) and \( z = 3 \).
1. If the surface can be described parametrically as
\[ \{(x(u, v), y(u, x), z(u, v))|(u, v) \in D\} \]
for some set \( D \) in \( uv\)-plane, set-up \( r = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \), and compute \( r_u, r_v \), then their cross product \( r_u \times r_v \), and finally its length \( |r_u \times r_v| \). Put
\[ dS = |r_u \times r_v| du dv \ . \]

On the other hand if the surface is given as \( z = f(x, y) \), with some description where it lives, figure out the “floor” (projection on the \( xy\)-plane), and put
\[ dS = \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \ dA \]

**Note:** The first method also works in the second case, just take \( x, y \) as the parameters and put \( x = x, y = y, z = f(x, y) \).

1. In this problem \( z^2 = x^2 + y^2 \) means \( z = \sqrt{x^2 + y^2} \), so it is possible to do it the second way. But it is a bit easier to use cylindrical coordinates and then the surface is \( z = r \), and the parametric representation is
\[ x = r \cos \theta, y = r \sin \theta, z = r \ , \]
where the parameters are \( r \) and \( \theta \). The region on the \( xy\)-plane below the surface is
\[ \{(r, \theta) | 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\} \ . \]

So
\[ r(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k} \ . \]
\[ r_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \mathbf{k} \ . \]
\[ r_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} + 0 \mathbf{k} \ . \]
\[ r_r \times r_\theta = -r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} + r \mathbf{k} \ , \]
(you do it!). Also
\[ |r_r \times r_\theta| = \sqrt{(-r \cos \theta)^2 + (-r \sin \theta)^2 + r^2} = \sqrt{2}r \ , \]
so \( dS = \sqrt{2}r \ dr \ d\theta \).
2. Set-up the integral

\[ \int \int_S F(x, y, z) \, dS \]

with the \( x, y, z \) replaced by their expressions in terms of the parameters, and \( S \) replaced by its description in terms of \( u, v \), and \( dS \) replaced by what you found in step 1.

\[ \int \int_S 2x^2 z^2 \, dS \]

where \( D \) is the region \( 1 \leq r \leq 3, \, 0 \leq \theta \leq 2\pi \). Converting it to an iterated integral, we get

\[ = 2\sqrt{2} \int_0^{2\pi} \int_1^3 r^5 \cos^2 \theta \, dr \, d\theta \]

3. Evaluate the iterated integral.

\[ = 2\sqrt{2} \int_0^{2\pi} \left[ \int_1^3 r^5 \cos^2 \theta \, dr \right] d\theta \]

\[ = \sqrt{2} \int_0^{2\pi} \left[ \int_1^3 r^5 \cos^2 \theta \, dr \right] d\theta \]

\[ = \sqrt{2} \int_0^{2\pi} 2 \cos^2 \theta \left[ \frac{r^6}{6} \right]_1^3 \, d\theta \]

\[ = \sqrt{2} \left( \frac{3^6 - 1^6}{6} \right) \cdot \int_0^{2\pi} 2 \cos^2 \theta \, d\theta \]

\[ = \sqrt{2} \left( \frac{364}{3} \right) \cdot \int_0^{2\pi} (1 + \cos(2\theta)) \, d\theta \]

\[ = \sqrt{2} \left( \frac{364}{3} \right) \cdot \left[ \theta + \frac{\sin(2\theta)}{2} \right]_0^{2\pi} = \sqrt{2} \frac{364}{3} \cdot 2\pi = \frac{728\sqrt{2}\pi}{3} \] \[ \text{Ans.:} \, \frac{728\sqrt{2}\pi}{3} \]

**Problem Type 16.5b**: Evaluate the surface integral \( \int \int_S F \cdot dS \) for the given vector field \( F \) and oriented surface \( S \).

\[ F(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \]

where \( S \) is the part of the surface \( z = g(x, y) \) that lies above some region \( D \), in the \( xy \)-plane and has upward orientation.

**Example Problem 16.5b**: Evaluate the surface integral \( \int \int_S F \cdot dS \) for the given vector field \( F \)
and oriented surface $S$. 

$$
\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}
$$

$S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ and has upward orientation.

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**Steps**

1. **Set-up the formula**

$$
\int \int_{S} \mathbf{F} \cdot d\mathbf{S} = \int \int_{D} \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA
$$

Be also sure to replace $z$ by $g(x, y)$.

**Example**

1. **Here** $g = 4 - x^2 - y^2$, 

$$
P = xy \quad Q = yz \quad R = zx
$$

Plugging everything in, our surface integral is 

$$
\int \int_{D} -xy(-2x) - yz(-2y) + xz) dA
$$

$$
= \int \int_{D} (2x^2y + (2y^2 + x)z) dA
$$

but since $z = 4 - x^2 - y^2$, this equals 

$$
= \int \int_{D} (2x^2y + 2y^2 + x(4 - x^2 - y^2)) dA
$$

$$
= \int \int_{D} (2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2) dA
$$

where $D$ is the square 

$$
\{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}
$$

2. **Looking at $D$ convert it into an iterated integral.**

$$
= \int_{0}^{1} \int_{0}^{1} (2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2) \, dx \, dy
$$

$$
= \int_{0}^{1} \int_{0}^{1} (-x^3 + x^2(-2y^2 + 2y) + x(-y^2 + 4) - 2y^4 + 8y^2) \, dx \, dy
$$

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3. Evaluate the iterated integral. 3. You do it! (it is rather tedious). The answer turns out to be $\frac{713}{180}$.

A Problem from a Previous Final: Evaluate the surface integral

$$\int \int_{S} \sqrt{3} x \, dS,$$

where $S$ is the triangular region with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

Ans. $\frac{1}{2}$. 