Problem Type 16.4a: Find an equation of the tangent plane to the given parametric surface at the specified point.

Example Problem 16.4a: Find an equation of the tangent plane to the given parametric surface at the specified point.

Steps

1. Set-up

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} , \]

and compute the partial derivatives w.r.t. \( u \) and w.r.t. \( v \):

\[ \mathbf{r}_u = x_u \mathbf{i} + y_u \mathbf{j} + z_u \mathbf{k} , \]
\[ \mathbf{r}_v = x_v \mathbf{i} + y_v \mathbf{j} + z_v \mathbf{k} . \]

Then plug-in the given values of \( u \) and \( v \).

Example

1. In this problem

\[ \mathbf{r} = u^2 \mathbf{i} + v^2 \mathbf{j} + uv \mathbf{k} , \]

We have

\[ \mathbf{r}_u = 2u \mathbf{i} + 0 \mathbf{j} + v \mathbf{k} , \]
\[ \mathbf{r}_v = 0 \mathbf{i} + 2v \mathbf{j} + u \mathbf{k} . \]

Now plug-in \( u = 1, v = 1 \) to get numerical vectors.

\[ \mathbf{r}_u(1,1) = 2 \mathbf{i} + 0 \mathbf{j} + 1 \mathbf{k} , \]
\[ \mathbf{r}_v(1,1) = 0 \mathbf{i} + 2 \mathbf{j} + 1 \mathbf{k} . \]

So at this point, \( \mathbf{r}_u = \langle 2,0,1 \rangle, \mathbf{r}_v = \langle 0,2,1 \rangle . \)
2. Find the cross-product $\mathbf{r}_u \times \mathbf{r}_v$. This is a vector **normal** to the tangent plane.

\[
\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 0 & 1 \\
0 & 2 & 1
\end{vmatrix}
= -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} .
\]

Or in $\langle \rangle$ notation
\[
\mathbf{N} = \langle -2, -2, 4 \rangle .
\]

3. Find the point $(x_0, y_0, z_0)$ by plugging into $x, y, z$ the specific values of $u$ and $v$ given in the problem. The desired equation of the tangent plane is
\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 .
\]
where $\mathbf{N} = \langle a, b, c \rangle$ and the point is $(x_0, y_0, z_0)$.

The point is $(1^2, 1^2, 1 \cdot 1) = (1, 1, 1)$.

The desired equation of the tangent plane is
\[
(-2)(x - 1) - 2(y - 1) + 4(z - 1) = 0 .
\]

Or, in expanded form
\[
-2x - 2y + 4z = 0 .
\]

Dividing by $-2$ (to make it nicer), we get:
\[
\text{Ans.:} \quad x + y - 2z = 0 .
\]

**A Problem from a previous Final:** Find an equation for the tangent plane to the parametric surface
\[
x = u^2 , \quad y = u + v , \quad z = v^2 ,
\]
at the point $(1, 1, 1)$. Simplify as much as you can!

\[
\text{Ans.:} \quad x - 2y + z = -2 .
\]

**Another Problem from a Previous Final:** Evaluate the surface integral
\[
\int \int_S \sqrt{3} x \, dS ,
\]
where $S$ is the triangular region with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

\[
\text{Ans.} \quad \frac{1}{2} .
\]