Problem Type 16.2a: Evaluate the line integral,
\[ \int_C f(x,y) \, ds , \]
where \( C \) is some curve that the problem gives you in parametric form, or you have to represent yourself (typically circles, line-segments, semicircles etc.).

Example Problem 16.2a: Evaluate the line integral,
\[ \int_C x^2 y \, ds , \]
where \( C \) is top half of the circle \( x^2 + y^2 = 9 \).

Steps

1. Find the parametric equation of the curve \((x(t), y(t))\), \(a \leq t \leq b\), unless it is given by the problem.

   1. The parametric equation of a circle of the form \( x^2 + y^2 = r^2 \) is
      \[ x = r \cos t , \ y = r \sin t \ . \]
      So in our case we have \( r = 3 \) and
      \[ x = 3 \cos t , \ y = 3 \sin t \ . \]
      Since it is the top half, \( t \) goes from 0 to \( \pi \), so \( 0 \leq t \leq \pi \).

2. Compute \( \sqrt{x'(t)^2 + y'(t)^2} \).

   2. \( x'(t) = -3 \sin t \), \( y'(t) = 3 \cos t \), so
      \[ \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} \]
      \[ = \sqrt{9 \sin^2 t + 9 \cos^2 t} = \sqrt{9 (\sin^2 t + \cos^2 t)} = \sqrt{9} = 3 \ . \]
3. The line integral is
\[ \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt \, . \]
Convert everything to the \(t\)-language and evaluate the \(t\)-integral from \(t = a\) to \(t = b\).

\[ \int_C x^2 y \, ds = \int_0^\pi (3 \cos^2 t) (3 \sin t) \cdot 3 \, dt = 81 \int_0^\pi \cos^2 t \sin t \, dt = 81 \left( \frac{-\cos^3 \pi}{3} \right) \bigg|_0^\pi = \frac{81(\cos^3 \pi - \cos^3 0)}{3} = 54 \, . \]
Ans.: 54.

**Problem Type 16.2b:** Evaluate the line integral
\[ \int_C P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz \, , \]
where \(C : x = x(t), y = y(t), z = z(t), a \leq t \leq b\).

**Example Problem 16.2b:** Evaluate the line integral
\[ \int_C y \, dx + x \, dy + x^2 y \sqrt{z} \, dz \, , \]
where \(C : x = t^3, y = t, z = t^2, 0 \leq t \leq 1\).

**Steps**

1. Get a (single variable) definite integral, in \(t\), from \(t = a\) to \(t = b\), by changing \(x, y, z\) to their expressions in terms of \(t\) and \(dx, dy, dz\) to \(x'(t)dt, y'(t)dt, z'(t)dt\), respectively,
\[ \int_C P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz \, . \]
\[ = \int_a^b [P(x(t), y(t), z(t)) x'(t) + Q(x(t), y(t), z(t)) y'(t) + R(x(t), y(t), z(t)) z'(t)] \, dt \, . \]

1. 
\[ \int_C y \, dx + x \, dy + x^2 y \sqrt{z} \, dz \]
\[ = \int_0^1 t(3t^2) \, dt + t^3 \, dt + (t^3)^2 t \sqrt{t^2} (2t) \, dt = \int_0^1 [4t^3 + 2t^9] \, dt \, . \]
2. Evaluate the $t$-integration.

\[
\int C F \cdot dr = t^4 + \frac{t^{10}}{5} \bigg|_0^1 = 1 + \frac{1}{5} - 0 = \frac{6}{5}.
\]

Ans.: $\frac{6}{5}$.

**Problem Type 16.2c**: Evaluate the line integral

\[
\int_C F \cdot dr,
\]

where $C$ is given by the vector function $\mathbf{r}(t)$.

\[
F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k,
\]

\[
\mathbf{r}(t) = x(t)i + y(t)j + z(t)k, \quad a \leq t \leq b.
\]

**Example Problem 16.2c**: Evaluate the line integral

\[
\int_C F \cdot dr,
\]

where $C$ is given by the vector function $\mathbf{r}(t)$.

\[
F(x,y,z) = yzi + xzj + xyk,
\]

\[
\mathbf{r}(t) = ti + t^2j + t^3k, \quad 0 \leq t \leq 2.
\]

**Steps**

1. The desired line-integral equals

\[
\int_C P \, dx + Q \, dy + R \, dz.
\]

Set-it up.

**Example**

1. Our integral is

\[
\int_C yz \, dx + xz \, dy + xy \, dz,
\]

where $x = t$, $y = t^2$, $z = t^3$, $0 \leq t \leq 2$. 
2. Evaluate this line integral like we did above (16.2b).

\[
\int_0^2 (t^2)(t^3) \, dt + (t)(t^3)(2t) \, dt + (t)(t^2)(3t^2) \, dt, \\
= \int_0^2 t^5 \, dt + 2t^5 \, dt + 3t^5 \, dt, \\
= \int_0^2 6t^5 \, dt = t^6 \bigg|_0^2 = 2^6 - 0^6 = 64.
\]

Ans.: 64.

A Problem from a previous Final

Let \( C \) be the line segment from \((0, 1)\) to \((3, 5)\), find \( \int_C 2xy \, ds \).

Ans.: 55.

Another Problem from a Previous Final

(a) (4 points) Compute the surface integral

\[
\int \int_S 8 \, dS, 
\]

where \( S \) is the sphere \((x - 1)^2 + (y + 4)^2 + (z - 9)^2 = 100\).

(b) (4 points) Compute the triple integral

\[
\int \int \int_E 30 \, dV, 
\]

where \( E \) is the ball \( \{(x, y, z) \mid (x - 1)^2 + (y + 4)^2 + (z - 9)^2 \leq 100\} \).

(c) (4 points) Compute the line integral

\[
\int_C 3 \, ds, 
\]

where \( C \) is the circumference of the region \( \{(x, y) \mid x^2 + y^2 \leq 4, \ y \geq 0\} \).

Ans.: \( 3200\pi, 40000\pi, 6\pi + 12 \).