Dr. Z's Math251 Handout #15.6 (2nd ed.) [Change of Variables in Multiple Integrals]

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Problem Type 16.5a: Find the Jacobian of the transformation

$$x = g(u,v,w) \quad , \quad y = h(u,v,w) \quad , \quad z = k(u,v,w).$$

Example Problem 16.5a: Find the Jacobian of the transformation

$$x = u^2 v \quad , \quad y = v^2 w \quad , \quad z = w^2 u.$$

Steps

1. Compute all the entries in the Jacobian matrix

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad .$$

2. Evaluate the determinant:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= (\frac{\partial x}{\partial u}) \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - (\frac{\partial x}{\partial v}) \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$+(\frac{\partial x}{\partial w}) \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

Example

1.

$$\begin{vmatrix} 2uv & u^2 & 0 \\ 0 & 2vw & v^2 \\ w^2 & 0 & 2uw \end{vmatrix} \quad .$$

2.

$$= 2uv \begin{vmatrix} 2vw & v^2 \\ 0 & 2wu \end{vmatrix} \quad - u^2 \begin{vmatrix} 0 & v^2 \\ w^2 & 2wu \end{vmatrix}$$

$$+0 \cdot \begin{vmatrix} 0 & 2vw \\ w^2 & 0 \end{vmatrix}$$

$$= 2uv[(2vw)(2uw)-0]-u^2[0-(v^2)(w^2)]+0$$

$$= 9u^2v^2w^2 \quad .$$

Ans.: $9u^2v^2w^2$.

Problem Type 15.6b: Use the given transformation to evaluate the integral

$$\int\int_R F(x,y)\, dA \quad ,$$

where R is the triangular region with vertices (p_1, p_2),(q_1, q_2), (r_1, r_2); $x = au + bv$, $y = cu + dv$.

Example Problem 15.6b: Use the given transformation to evaluate the integral

$$\int\int_R (x+y)\, dA \quad ,$$

where R is the triangular region with vertices $(0,0)$,$(2,1)$, $(1,2)$; $x = 2u + v$, $y = u + 2v$.

Steps

1. Figure out the region in the uv-plane that gets transformed. Since a triangle goes to a triangle, we need to find the 3 vertices. Solve for u, v in terms of x, y and find the three points. Call the new triangle R'.

Example

1. Since $x = 2u + v$, $y = u + 2v$, when $(x, y) = (0, 0)$ $u = 0, v = 0$ so the point $(0, 0)$ goes to the point $(0, 0)$. When $(x, y) = (1, 2)$, we have to solve the system $1 = 2u+v, 2 = u+2v$ giving us $u = 0, v = 1$ so $(1, 2)$ goes to $(0, 1)$. Similarly, $(2, 1)$ goes to $(1, 0)$. So the region in the uv-plane is the far simpler triangle whose vertices are $(0, 0), (1, 0), (0, 1)$. Let's call this region R'.

2. Find the Jacobian of the transformation. In this case of a so-called linear transformation, the Jacobian is simply $ad-bc$. Also express $F(x, y)$ in terms of (u, v) using the transformation.

$$\int\int_R F(x,y)\, dA =$$

$$\int\int_{R'} F(au+bv, cu+dv)|(ad-bc)|\, dA \quad .$$

(Note that one must take the **absolute value** of the Jacobian)

2. The Jacobian is $(2)(2) - (1)(1) = 3$, so

$$\int\int_R (x+y)\, dA = \int\int_{R'} (2u+v+u+2v)\cdot|3|\, dA =$$

$$9\int\int_{R'} (u+v)\, dA \quad .$$

3. Draw the region (in this case triangle) in the uv- plane and express it as a type I (or type II) region. Then set-up the appropriate iterated integral, by deciding on the **main road** and the **side streets**.

3. The region is the triangle bounded by the axes and the line $u+v=1$. It can be written as

$$\{(u,v)\,|\,0 \le u \le 1\,,\,0 \le v \le 1-u\,\} \quad .$$

Our area-integral is thus equal to the iterated integral

$$9\int_0^1 \int_0^{1-u} (u+v)\,dv\,du \quad .$$

The inner integral is

$$\int_0^{1-u} (u+v)\,dv = uv + \frac{v^2}{2}\Big|_0^{1-u}$$

$$= u(1-u) + \frac{(1-u)^2}{2} = (1-u^2)/2 \quad ,$$

and the whole integral is

$$\frac{9}{2}\int_0^1 (1-u^2)\,du = \frac{9}{2}\left[u - \frac{u^3}{3}\right]\Big|_0^1 = \frac{9}{2}\cdot\frac{2}{3} = 3 \quad .$$

Ans.: 3.

A Problem from a previous Final

Find the Jacobian of the transformation

$$x = u+v+w \quad , \quad y = u^2+v^2+w^2 \quad , \quad z = u^3+v^3+w^3 \quad .$$

Simplify as much as you can!

Ans.: $6(vw^2 - v^2w - uw^2 + u^2w + uv^2 - u^2v)$.

Another Problem from a Previous Final

Use the transformation

$$x = 2u+v \quad , \quad y = u+2v \quad ,$$

to evaluate the integral

$$\int\int_R (2x-y)\,dA$$

where R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$.

Ans.: $\frac{3}{2}$.